X-863-73-112
PREPRINT

NASA TM X- 70517

RELATIVE SIDEBAND AMPLITUDES VS. MODULATION INDEX FOR COMMON FUNCTIONS USING FREQUENCY AND PHASE MODULATION

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NOVEMBER 1973

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

(NASA-TM-X-70517) RELATIVE SIDEBAND AMPLITUDES VERSUS MODULATION INDEX FOR COMMON FUNCTIONS USING FREQUENCY AND PHASE MODULATION (NASA) 48 p HC $4.50 CSCL 20N G3/07
N74-11[illegible]38
Unclas 22810

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ABSTRACT

The equations defining the amplitude of sidebands resulting from either frequency modulation or phase modulation by either square-wave, sine-wave, sawtooth or triangular modulating functions are presented. Spectral photographs and computer-generated tables of modulation index vs. relative sideband amplitudes are also included.

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INTRODUCTION

An important consideration in the design and testing of a communications system is the relative amplitude of sideband components resulting from modulation of a radio frequency (RF) carrier. This spectral information is useful in compatibility testing to determine modulation parameters and can also be used to precisely select optimum receiver bandwidths. This document is intended to be used as a handbook for engineers involved in such activities. The majority of the information is presented in tabular form (i.e. modulation index vs. relative sideband amplitudes). Spectrum photographs are included for illustrative purposes. Theoretical background material is given for completeness and convenience.

Appendix A contains the square-wave FM and PM analysis and develops the equations for the sideband amplitudes.

Appendix B extends the PM analysis for the sawtooth and triangular modulating function. This illustrates the generality of the analysis technique.

SQUARE-WAVE FREQUENCY MODULATION (FM)

The general expression (derived in Appendix A) for the amplitude of the nth sideband resulting from square-wave frequency modulation of a unity-amplitude RF carrier is

$$|C_n| = \left|\frac{\left(\frac{2\beta}{\pi}\right)\sin\left[(\beta - n)\frac{\pi}{2}\right]}{\beta^2 - n^2}\right|, \quad \beta \neq n \tag{1}$$

Taking the limit as $\beta \to n$

$$|C_n| = \frac{1}{2}, \quad \beta = n; \tag{2}$$

where $\beta \equiv$ modulation index in radians $= \dfrac{\text{RF carrier deviation } (\Delta f)}{\text{modulating frequency (fm)}}$,

n = sideband order (0, 1, 2, . . .),

$|C_n|$ = absolute value of n^{th} order sideband.

The expression for the residual carrier amplitude is obtained by setting n = 0 in Equation (1):

$$|C_0| = \left|\frac{\sin\left(\beta\frac{\pi}{2}\right)}{\beta\frac{\pi}{2}}\right|. \tag{3}$$

Note that the right-hand side of Equation (3) has the form of sin x/x, which has a limit value of 1 as $x \to 0$; for the no-modulation case, the carrier amplitude is equal to 1.

The expression for the first sideband amplitude term (n = 1) is

$$|C_1| = \left(\frac{2\beta}{\pi}\right)\frac{\sin\left[(\beta - 1)\frac{\pi}{2}\right]}{(\beta^2 - 1)}. \tag{4}$$

A convenient ratio for measurement purposes is the ratio of the residual carrier amplitude to the first order sideband amplitude. This ratio is given by

$$\frac{|C_0|}{|C_1|} = \left(\frac{\beta^2 - 1}{\beta^2}\right) \tan\left(\beta\frac{\pi}{2}\right) \tag{5}$$

This result can be used to determine the relative difference between the residual carrier level and the first sideband level on a spectrum analyzer. Expressions for the higher-order sidebands are given in Appendix A. Table 1 contains the solution of Equation (1) for the first five sidebands for modulation indices up to 10; all sideband amplitudes are normalized to unity as follows:

Relative sideband amplitude = $20 \log_{10} (1/|C_n|) = -20 \log_{10} |C_n|$.

Table 2 illustrates the use of Table 1. Unmodulated carrier power level: 1 watt (30 dbm); FM mod index = 1.

The following salient properties of square-wave frequency modulation should be noted:

(a) The residual carrier is zero for all even-integer modulation indices;

(b) When β is an odd integer, the odd-ordered sidebands equal zero (except the β = n sideband);

(c) When β is an even integer, the even-order sidebands equal zero (except the β = n sideband);

(d) When β = n, one-half the spectral energy resides in the β = n sideband; the remaining spectral energy is contained in the residual carrier and other sidebands;

(e) Except for the special cases noted previously, all sidebands are present in the FM spectrum although only odd harmonics exist in the modulating signal.

Figure 1 illustrates the basic test setup used to obtain spectrum photographs. These photographs are shown in Figure 2. The modulation indices were chosen to demonstrate the spectrum characteristics discussed previously. A decibel scale (photograph (e)) is provided for measuring amplitudes from the pictures. This scale is valid for all RF spectrum photographs in Figures 2, 3, 4, B1 and B4. For each modulation index, the corresponding spectrum resulting from the equivalent pseudo-random (PR) modulating signal is shown.* This is done both because the PR spectrum is a good approximation to the actual spacecraft data spectrum, and also to illustrate the problem in the setting of modulation index for complex signals.

* Equivalent PR signal is defined as follows: Assume the square wave to be generated by using an alternating 1, 0 NRZ-L pattern. The PR code is then clocked at the same square-wave rate, but with its own unique 1, 0 pattern.

Table 1
Square-Wave Frequency Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)
0.10	-0.04	-23.94	23.91	-52.06	-43.11	-64.11	-51.99
0.20	-0.14	-17.98	17.84	-40.06	-37.38	-52.16	-46.28
0.30	-0.32	-14.56	14.24	-33.08	-34.38	-45.27	-43.31
0.40	-0.58	-12.21	11.63	-28.18	-32.65	-40.49	-41.63
0.50	-0.91	-10.45	9.54	-24.43	-31.79	-36.90	-40.82
0.60	-1.33	-9.10	7.77	-21.42	-31.71	-34.08	-40.81
0.70	-1.83	-8.03	6.20	-18.93	-32.48	-31.84	-41.67
0.80	-2.42	-7.19	4.77	-16.82	-34.51	-30.02	-43.79
0.90	-3.11	-6.53	3.41	-15.02	-39.22	-28.58	-48.62
1.00	-3.92	-6.02	2.10	-13.46		-27.44	
1.10	-4.86	-5.65	0.79	-12.11	-37.04	-26.60	-46.74
1.20	-5.94	-5.41	-0.53	-10.94	-30.11	-26.04	-39.98
1.30	-7.20	-5.28	-1.92	-9.92	-25.78	-25.76	-35.85
1.40	-8.69	-5.26	-3.42	-9.03	-22.57	-25.79	-32.87
1.50	-10.45	-5.35	-5.11	-8.27	-20.00	-26.18	-30.55
1.60	-12.62	-5.54	-7.08	-7.62	-17.86	-27.02	-28.70
1.70	-15.39	-5.85	-9.55	-7.08	-16.04	-28.52	-27.21
1.80	-19.23	-6.26	-12.97	-6.63	-14.46	-31.13	-26.01
1.90	-25.61	-6.79	-18.82	-6.28	-13.09	-36.32	-25.06
2.00		-7.44		-6.02	-11.88		-24.35
2.10	-26.48	-8.24	-18.24	-5.85	-10.82	-34.87	-23.86
2.20	-20.97	-9.20	-11.77	-5.76	-9.89	-28.23	-23.60
2.30	-18.02	-10.34	-7.68	-5.76	-9.08	-24.14	-23.58
2.40	-16.14	-11.71	-4.43	-5.84	-8.37	-21.14	-23.84
2.50	-14.89	-13.38	-1.51	-6.02	-7.76	-18.75	-24.43
2.60	-14.06	-15.45	1.38	-6.28	-7.24	-16.78	-25.46
2.70	-13.55	-18.13	4.58	-6.64	-6.81	-15.10	-27.12
2.80	-13.30	-21.88	8.58	-7.10	-6.47	-13.65	-29.87
2.90	-13.28	-28.18	14.91	-7.67	-6.20	-12.39	-35.18
3.00	-13.46			-8.36	-6.02	-11.28	
3.10	-13.86	-28.91	15.05	-9.18	-5.92	-10.31	-33.95
3.20	-14.46	-23.33	8.87	-10.16	-5.89	-9.46	-27.40
3.30	-15.30	-20.32	5.02	-11.32	-5.94	-8.72	-23.40
3.40	-16.39	-18.38	1.99	-12.70	-6.07	-8.08	-20.48
3.50	-17.81	-17.07	-0.74	-14.38	-6.29	-7.53	-18.16
3.60	-19.66	-16.19	-3.47	-16.46	-6.59	-7.07	-16.25
3.70	-22.15	-15.63	-6.52	-19.14	-6.98	-6.69	-14.63
3.80	-25.72	-15.33	-10.39	-22.90	-7.47	-6.39	-13.24
3.90	-31.86	-15.26	-16.60	-29.21	-8.07	-6.17	-12.02
4.00		-15.40			-8.78	-6.02	-10.97
4.10	-32.29	-15.75	-16.54	-29.93	-9.63	-5.95	-10.04
4.20	-26.59	-16.32	-10.27	-24.35	-10.62	-5.95	-9.23
4.30	-23.45	-17.11	-6.34	-21.33	-11.80	-6.04	-8.53
4.40	-21.41	-18.17	-3.24	-19.40	-13.20	-6.20	-7.92

Table 1 (continued)
Square-Wave Frequency Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)
4.50	-20.00	-19.56	-0.44	-18.09	-14.89	-6.44	-7.4[illegible]
4.60	-19.02	-21.37	2.35	-17.20	-16.98	-6.76	-6.97
4.70	-18.37	-23.82	5.45	-16.63	-19.68	-7.18	-6.62
4.80	-17.98	-27.36	9.38	-16.33	-23.45	-7.68	-6.34
4.90	-17.83	-33.47	15.64	-16.25	-29.76	-8.30	-6.14
5.00	-17.90			-16.39		-9.03	-6.02
5.10	-18.18	-33.85	15.67	-16.73	-30.50	-9.89	-5.97
5.20	-18.68	-28.12	9.44	-17.29	-24.93	-10.90	-6.00
5.30	-19.41	-24.95	5.54	-18.08	-21.91	-12.09	-6.10
5.40	-20.41	-22.88	2.47	-19.13	-19.98	-13.50	-6.27
5.50	-21.74	-21.45	-0.29	-20.51	-18.67	-15.20	-6.53
5.60	-23.50	-20.45	-3.06	-22.32	-17.79	-17.30	-6.87
5.70	-25.90	-19.77	-6.13	-24.76	-17.22	-20.01	-7.30
5.80	-29.39	-19.36	-10.03	-28.29	-16.92	-23.78	-7.82
5.90	-35.45	-19.19	-16.26	-34.39	-16.85	-30.11	-8.45
6.00		-19.24			-16.99		-9.19
6.10	-35.74	-19.50	-16.24	-34.75	-17.33	-30.86	-10.06
6.20	-29.97	-19.98	-9.99	-29.02	-17.89	-25.30	-11.08
6.30	-26.77	-20.69	-6.08	-25.85	-18.68	-22.29	-12.28
6.40	-24.66	-21.67	-2.99	-23.77	-19.73	-20.36	-13.70
6.50	-23.19	-22.98	-0.21	-22.33	-21.11	-19.06	-15.41
6.60	-22.15	-24.73	2.57	-21.32	-22.92	-18.18	-17.52
6.70	-21.45	-27.11	5.66	-20.64	-25.36	-17.62	-20.23
6.80	-21.01	-30.58	9.57	-20.22	-28.89	-17.32	-24.02
6.90	-20.81	-36.63	15.82	-20.04	-34.99	-17.25	-30.34
7.00	-20.82			-20.08		-17.39	
7.10	-21.06	-36.89	15.83	-20.34	-35.35	-17.74	-31.11
7.20	-21.50	-31.10	9.60	-20.81	-29.61	-18.30	-25.55
7.30	-22.19	-27.88	5.69	-21.51	-26.44	-19.09	-22.55
7.40	-23.15	-25.76	2.61	-22.49	-24.36	-20.15	-20.63
7.50	-24.43	-24.28	-0.16	-23.79	-22.92	-21.53	-19.33
7.60	-26.15	-23.23	-2.93	-25.53	-21.91	-23.34	-18.45
7.70	-28.51	-22.51	-6.00	-27.90	-21.22	-25.78	-17.90
7.80	-31.96	-22.06	-9.91	-31.37	-20.81	-29.31	-17.60
7.90	-37.99	-21.84	-16.15	-37.41	-20.63	-35.42	-17.54
8.00		-21.85			-20.67		-17.68
8.10	-38.21	-22.07	-16.14	-37.66	-20.92	-35.78	-18.03
8.20	-32.40	-22.50	-9.89	-31.87	-21.39	-30.04	-18.60
8.30	-29.16	-23.18	-5.98	-28.64	-22.09	-26.87	-19.39
8.40	-27.02	-24.11	-2.90	-26.52	-23.06	-24.79	-20.45
8.50	-25.52	-25.40	-0.12	-25.03	-24.37	-23.35	-21.83
8.60	-24.45	-27.11	2.66	-23.97	-26.10	-22.34	-23.64
8.70	-23.72	-29.46	5.74	-23.24	-28.47	-21.65	-26.09
8.80	-23.25	-32.90	9.65	-22.79	-31.94	-21.24	-29.63
8.90	-23.02	-38.91	15.90	-22.57	-37.98	-21.06	-35.73
9.00	-23.01			-22.57		-21.10	
9.10	-23.21	-39.11	15.90	-22.78	-38.22	-21.35	-36.10
9.20	-23.63	-33.30	9.66	-23.21	-32.42	-21.81	-30.36
9.30	-24.29	-30.05	5.76	-23.88	-29.20	-22.52	-27.19
9.40	-25.23	-27.90	2.68	-24.82	-27.07	-23.49	-25.11
9.50	-26.49	-26.39	-0.10	-26.09	-25.57	-24.79	-23.67
9.60	-28.18	-25.31	-2.87	-27.80	-24.52	-26.53	-22.66
9.70	-30.52	-24.57	-5.95	-30.14	-23.79	-28.90	-21.98
9.80	-33.95	-24.09	-9.86	-33.58	-23.33	-32.36	-21.56
9.90	-39.95	-23.85	-16.09	-39.59	-23.11	-38.40	-21.38
10.00		-23.84			-23.10		-21.42

Table 2
Sample Sideband Energies

Spectrum Element	db Value From Table 1	dbm Power Level	Single Sideband Power (watts)	Total Sideband Power (watts)
Residual carrier level	- 3.92	26.08	0.4055	0.4055
First sideband level[a]	- 6.02	23.98	0.250	0.500
Residual carrier/First sideband	2.10	-	-	-
Second sideband level	-13.46	16.54	0.045	0.090
Third sideband level[b]	Not Present	∞	0.000	0.000
Fourth sideband level	-27.44	2.56	0.0018	0.0036
Fifth sideband level[b]	Not Present	∞	0.000	0.000
TOTAL POWER				0.9991[c]

[a] When $\beta = n$, half of the total spectral energy resides in that pair of sidebands (1/4 in each sideband).

[b] When β = odd integer, all odd sidebands are zero except the $\beta = n$ sideband.

[c] The remaining power resides in the higher-order sidebands.

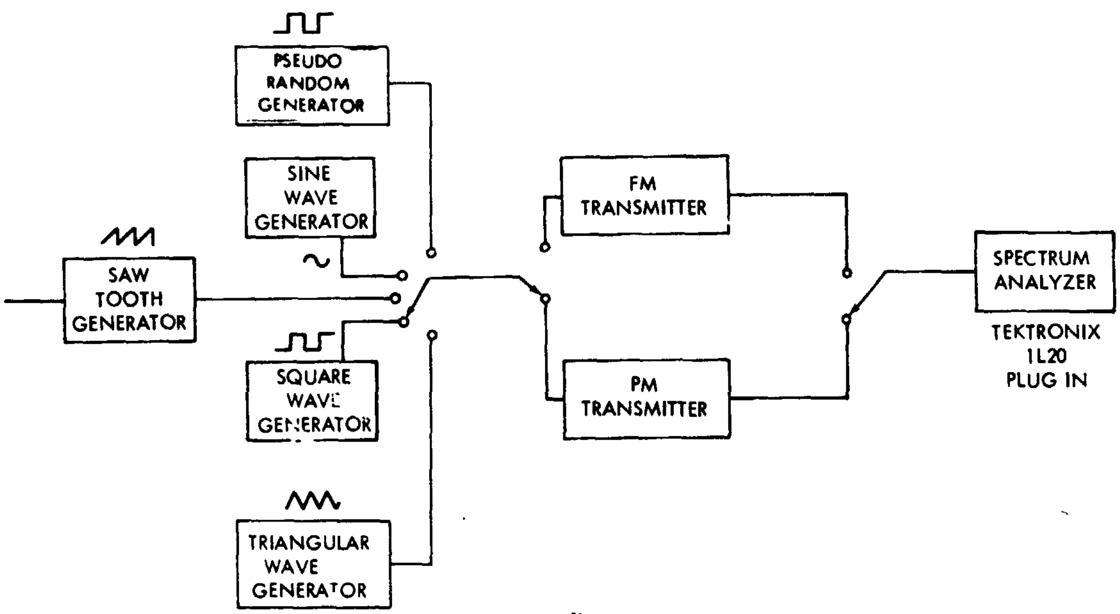


Figure 1. Basic Test Configuration Used to Obtain Spectra

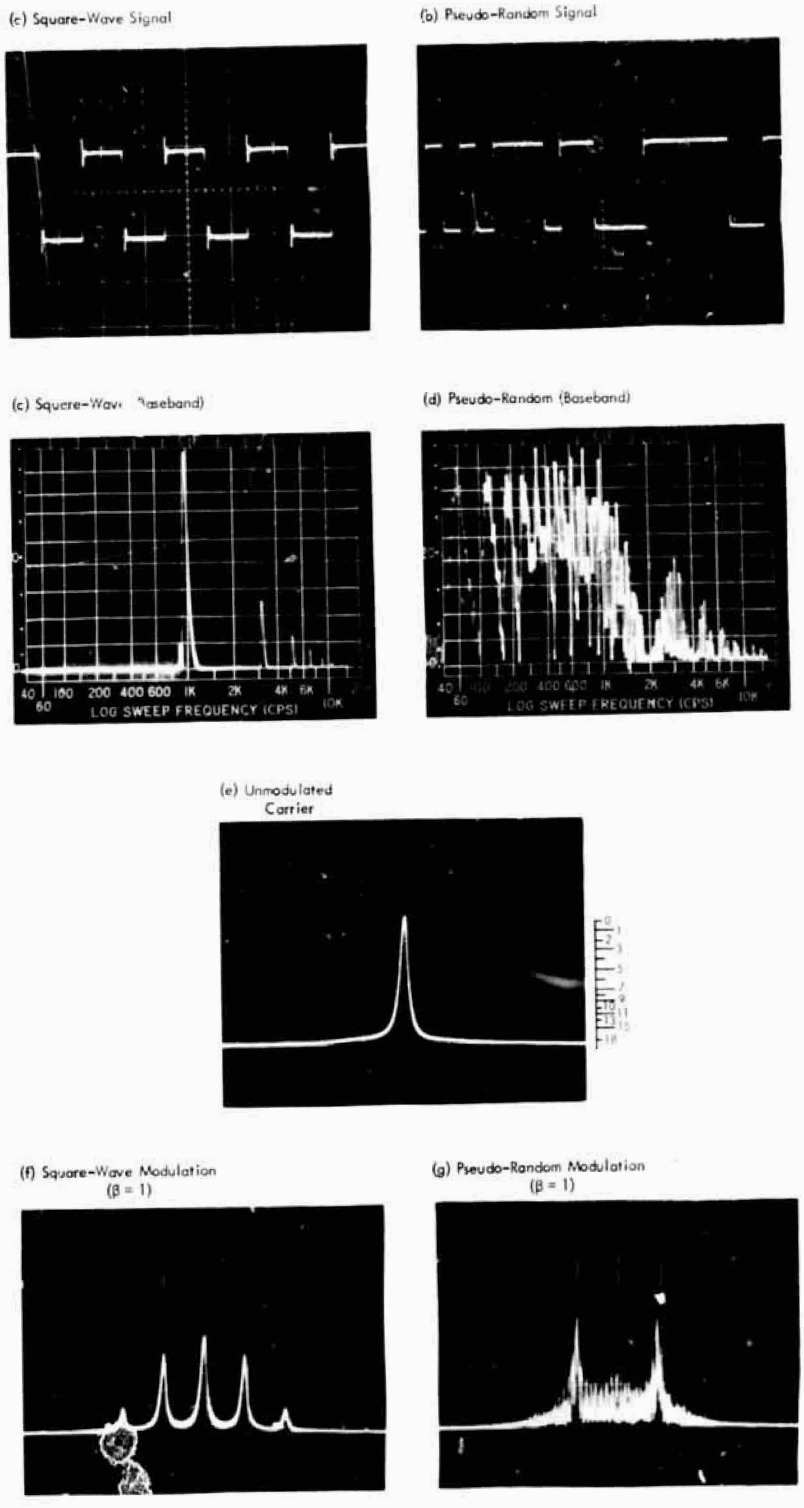


Figure 2. Frequency Modulation Spectra (Square-Wave and Pseudo-Random)

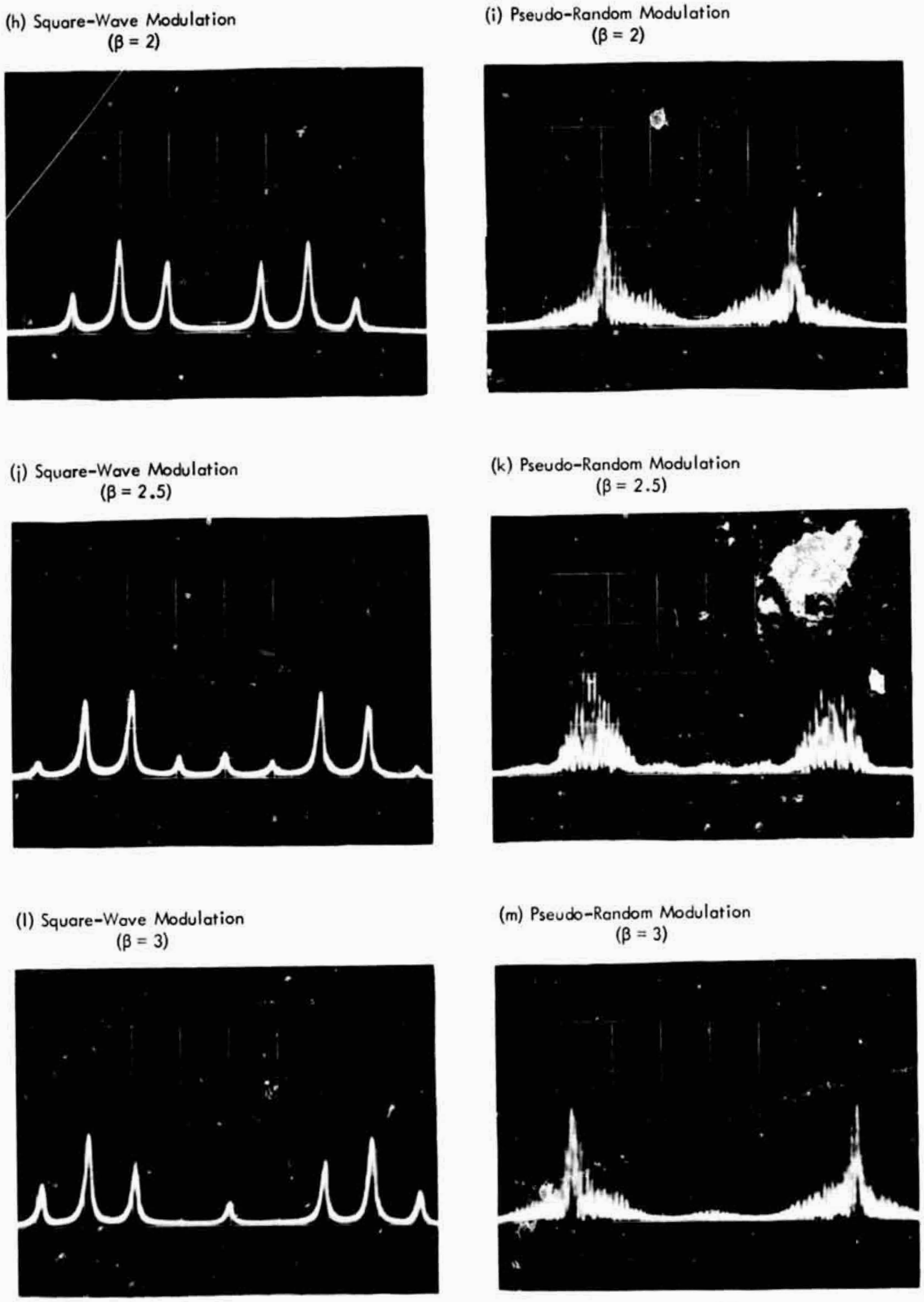
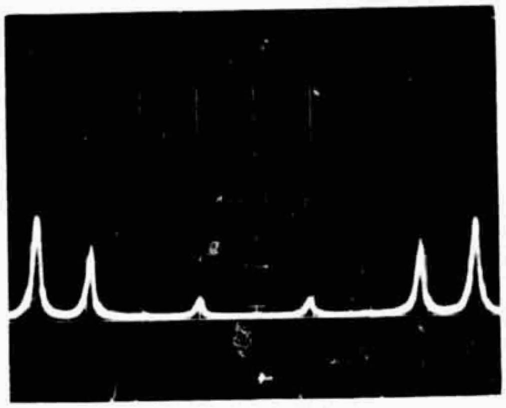
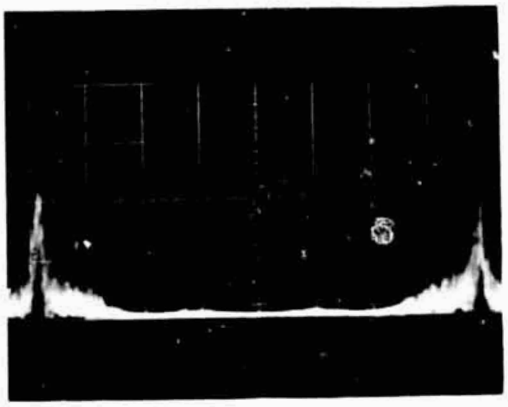


Figure 2(continued). Frequency Modulation Spectra (Square-Wave and Pseudo-Random)

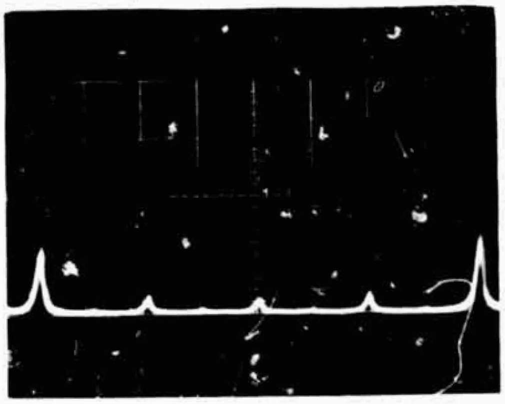
(n) Square-Wave Modulation
($\beta = 4$)



(o) Pseudo-Random Modulation
($\beta = 4$)



(p) Square-Wave Modulation
($\beta = 5$)



(q) Pseudo-Random Modulation
($\beta = 5$)

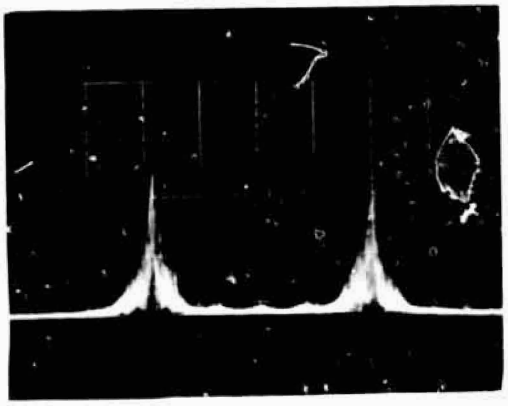


Figure 2(continued). Frequency Modulation Spectra (Square-Wave and Pseudo-Random)

SQUARE-WAVE PHASE MODULATION (PM)

Appendix A contains the derivation of the equations for the sideband amplitudes for this modulation case. Table 3 contains the solution of these equations for modulation indices up to 3 radians; this is the normal range of interest because most PM schemes require a low index to ensure adequate residual carrier power for phase lock demodulation and tracking. The significant PM spectral properties are:

(a) the PM spectrum is not the same as the FM spectrum, even for the same modulation index;

(b) no even-order sidebands are present in the resultant spectrum;

(c) the residual carrier goes to zero at a modulation index of $\pi/2$ (or any multiple of $\pi/2$) radians.

In general, the square-wave PM process can be viewed simply as amplitude modulation of the RF carrier by the modulating spectrum. If one considers the baseband spectrum as double-sided (i.e., containing both positive and negative image frequency components), then this spectrum is simply translated up in frequency symmetrically placed about the RF carrier. Only those frequency components existing in the modulating spectrum will be present in the PM spectrum. This point can be illustrated by writing the expression for the PM wave as

$$f(t) = A\cos(\omega_c t + \beta C(t)), \tag{6}$$

where

- A = amplitude of signal
- ω_c = radian carrier frequency
- $C(t)$ = modulating function
- β = peak phase deviation of carrier frequency in radians.

The modulating signal is restricted to values of ±1. This representation is valid for the PR signal or any binary function. Expanding Equation (6) yields

$$f(t) = A\cos\omega_c t\cos\beta C(t) - A\sin\omega_c t\sin\beta C(t). \tag{7}$$

Allowing $C(t) = \pm 1$, then

$$\cos\beta C(t) \rightarrow \cos\beta$$

$$\sin\beta C(t) \rightarrow C(t)\sin\beta. \tag{8}$$

and so

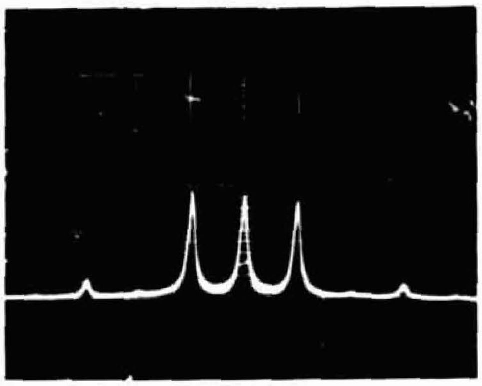
$$f(t) = A \cos \omega_c t \cos \beta - AC(t) \sin \beta \sin \omega_c t. \tag{9}$$

The first term in Equation (9) is simply the in-phase carrier term.

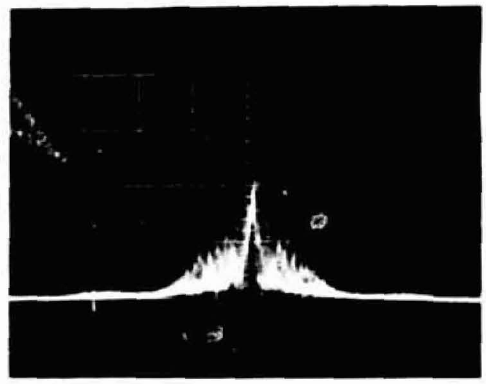
The second term has the form of amplitude modulation of the quadrature carrier component by $C(t) \sin \beta$. As previously stated, this means that the baseband spectrum of the digital signal is translated in frequency symmetrically about the RF carrier frequency. The spectrum pictures are shown in Figure 3.

For modulating signals other than the square-wave binary type, a unique analysis must be performed. This is done in Appendix B for a sawtooth and triangular function. Theoretically, the analysis can be done for any signal that is expressable by an equation; however, the mathematics becomes rather complex for other than simple functions.

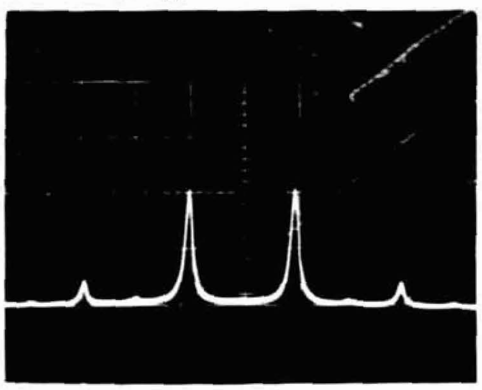
(a) Square-Wave Modulation
($\beta = 1$)



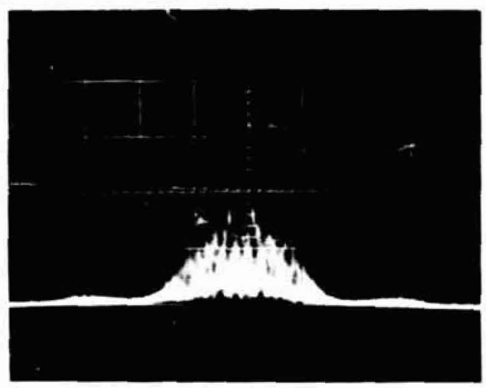
(b) Pseudo-Random Modulation
($\beta = 1$)



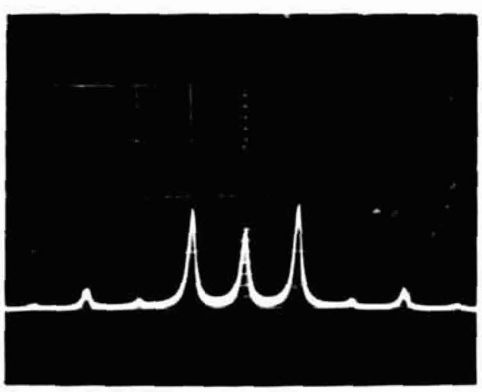
(c) Square-Wave Modulation
($\beta = \frac{\pi}{2}$)



(d) Pseudo-Random Modulation
($\beta = \frac{\pi}{2}$)



(e) Square-Wave Modulation
($\beta = 2$)



(f) Pseudo-Random Modulation
($\beta = 2$)

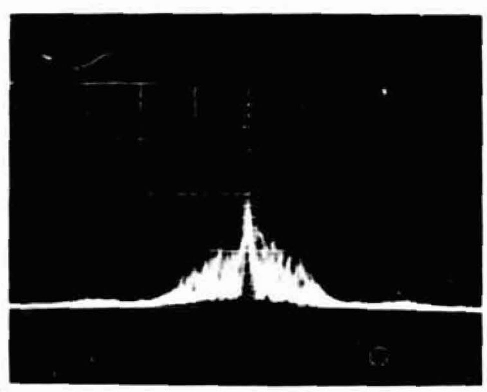


Figure 3. Phase Modulation Spectra (Square-Wave and Pseudo-Random)
NOTE. Unmodulated carrier reference and Modulation

Table 3
Square-Wave Phase Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C3 (DB)	C5 (DB)
0.10	-0.04	-23.93	23.89	-33.47	-37.91
0.11	-0.05	-23.11	23.05	-32.65	-37.09
0.12	-0.06	-22.36	22.29	-31.90	-36.33
0.13	-0.07	-21.66	21.59	-31.21	-35.64
0.14	-0.09	-21.02	20.94	-30.57	-35.00
0.15	-0.10	-20.43	20.33	-29.97	-34.41
0.16	-0.11	-19.87	19.76	-29.42	-33.85
0.17	-0.13	-19.35	19.22	-28.89	-33.33
0.18	-0.14	-18.86	18.72	-28.40	-32.84
0.19	-0.16	-18.40	18.24	-27.94	-32.37
0.20	-0.17	-17.96	17.78	-27.50	-31.93
0.21	-0.19	-17.54	17.34	-27.08	-31.52
0.22	-0.21	-17.14	16.93	-26.68	-31.12
0.23	-0.23	-16.76	16.53	-26.30	-30.74
0.24	-0.25	-16.40	16.14	-25.94	-30.38
0.25	-0.27	-16.05	15.78	-25.59	-30.03
0.26	-0.30	-15.72	15.42	-25.26	-29.70
0.27	-0.32	-15.40	15.08	-24.94	-29.38
0.28	-0.35	-15.09	14.74	-24.63	-29.07
0.29	-0.37	-14.79	14.42	-24.33	-28.77
0.30	-0.40	-14.51	14.11	-24.05	-28.49
0.31	-0.42	-14.23	13.81	-23.77	-28.21
0.32	-0.45	-13.96	13.51	-23.51	-27.94
0.33	-0.48	-13.71	13.22	-23.25	-27.69
0.34	-0.51	-13.46	12.94	-23.00	-27.44
0.35	-0.54	-13.21	12.67	-22.76	-27.19
0.36	-0.58	-12.98	12.40	-22.52	-26.96
0.37	-0.61	-12.75	12.14	-22.30	-26.73
0.38	-0.64	-12.53	11.89	-22.07	-26.51
0.39	-0.68	-12.32	11.64	-21.86	-26.30
0.40	-0.71	-12.11	11.40	-21.65	-26.09
0.41	-0.75	-11.91	11.16	-21.45	-25.89
0.42	-0.79	-11.71	10.92	-21.25	-25.69
0.43	-0.83	-11.52	10.69	-21.06	-25.50
0.44	-0.87	-11.33	10.46	-20.87	-25.31
0.45	-0.91	-11.15	10.24	-20.69	-25.13
0.46	-0.95	-10.97	10.02	-20.51	-24.95
0.47	-1.00	-10.80	9.80	-20.34	-24.78
0.48	-1.04	-10.63	9.59	-20.17	-24.61
0.49	-1.09	-10.46	9.38	-20.01	-24.44
0.50	-1.13	-10.30	9.17	-19.85	-24.28
0.51	-1.18	-10.15	8.96	-19.69	-24.13
0.52	-1.23	-9.99	8.76	-19.54	-23.97
0.53	-1.28	-9.84	8.56	-19.39	-23.82

Table 3 (continued)
Square-Wave Phase Modulation: Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C3 (DB)	C5 (DB)
0.54	-1.33	-9.70	8.36	-19.24	-23.68
0.55	-1.39	-9.55	8.17	-19.10	-23.53
0.56	-1.44	-9.41	7.97	-18.96	-23.39
0.57	-1.49	-9.28	7.78	-18.82	-23.25
0.58	-1.55	-9.14	7.59	-18.68	-23.12
0.59	-1.61	-9.01	7.40	-18.55	-22.99
0.60	-1.67	-8.88	7.22	-18.42	-22.86
0.61	-1.73	-8.76	7.03	-18.30	-22.74
0.62	-1.79	-8.63	6.85	-18.18	-22.61
0.63	-1.85	-8.51	6.66	-18.06	-22.49
0.64	-1.92	-8.40	6.48	-17.94	-22.38
0.65	-1.98	-8.28	6.30	-17.82	-22.26
0.66	-2.05	-8.17	6.12	-17.71	-22.15
0.67	-2.12	-8.06	5.94	-17.60	-22.04
0.68	-2.19	-7.95	5.76	-17.49	-21.93
0.69	-2.26	-7.84	5.59	-17.38	-21.82
0.70	-2.33	-7.74	5.41	-17.28	-21.72
0.71	-2.40	-7.64	5.23	-17.18	-21.61
0.72	-2.48	-7.54	5.06	-17.08	-21.51
0.73	-2.55	-7.44	4.88	-16.98	-21.42
0.74	-2.63	-7.34	4.71	-16.88	-21.32
0.75	-2.71	-7.25	4.53	-16.79	-21.23
0.76	-2.80	-7.15	4.36	-16.70	-21.13
0.77	-2.88	-7.06	4.19	-16.61	-21.04
0.78	-2.96	-6.98	4.01	-16.52	-20.95
0.79	-3.05	-6.89	3.84	-16.43	-20.87
0.80	-3.14	-6.80	3.66	-16.35	-20.78
0.81	-3.23	-6.72	3.49	-16.26	-20.70
0.82	-3.32	-6.64	3.32	-16.18	-20.62
0.83	-3.42	-6.56	3.14	-16.10	-20.54
0.84	-3.51	-6.48	2.97	-16.02	-20.46
0.85	-3.61	-6.40	2.79	-15.94	-20.38
0.86	-3.71	-6.33	2.62	-15.87	-20.31
0.87	-3.81	-6.25	2.44	-15.79	-20.23
0.88	-3.92	-6.18	2.26	-15.72	-20.16
0.89	-4.02	-6.11	2.09	-15.65	-20.09
0.90	-4.13	-6.04	1.91	-15.58	-20.02
0.91	-4.24	-5.97	1.73	-15.51	-19.95
0.92	-4.35	-5.90	1.55	-15.45	-19.88
0.93	-4.47	-5.84	1.37	-15.38	-19.82
0.94	-4.59	-5.77	1.19	-15.32	-19.75
0.95	-4.71	-5.71	1.01	-15.25	-19.69
0.96	-4.83	-5.65	0.82	-15.19	-19.63
0.97	-4.95	-5.59	0.64	-15.13	-19.57
0.98	-5.08	-5.53	0.45	-15.07	-19.51
0.99	-5.21	-5.47	0.26	-15.02	-19.45
1.00	-5.35	-5.42	0.07	-14.96	-19.40
1.01	-5.48	-5.36	-0.12	-14.90	-19.34
1.02	-5.62	-5.31	-0.32	-14.85	-19.29
1.03	-5.77	-5.26	-0.51	-14.80	-19.23

Table 3 (continued)
Square-Wave Phase Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C3 (DB)	C5 (DB)
1.04	-5.91	-5.20	-0.71	-14.75	-19.18
1.05	-6.06	-5.15	-0.91	-14.70	-19.13
1.06	-6.22	-5.10	-1.11	-14.65	-19.08
1.07	-6.37	-5.06	-1.32	-14.60	-19.04
1.08	-6.53	-5.01	-1.52	-14.55	-18.99
1.09	-6.70	-4.96	-1.73	-14.51	-18.94
1.10	-6.87	-4.92	-1.95	-14.46	-18 70
1.11	-7.04	-4.87	-2.16	-14.42	-18. 5
1.12	-7.22	-4.83	-2.38	-14.37	-18.81
1.13	-7.40	-4.79	-2.61	-14.33	-18.77
1.14	-7.58	-4.75	-2.83	-14.29	-18.73
1.15	-7.78	-4.71	-3.07	-14.25	-18.69
1.16	-7.97	-4.67	-3.30	-14.21	-18.65
1.17	-8.18	-4.64	-3.54	-14.18	-18.61
1.18	-8.38	-4.60	-3.78	-14.14	-18.58
1.19	-8.60	-4.56	-4.03	-14.11	-18.54
1.20	-8.82	-4.53	-4.29	-14.07	-18.51
1.21	-9.04	-4.50	-4.55	-14.04	-18.48
1.22	-9.28	-4.46	-4.81	-14.01	-18.44
1.23	-9.52	-4.43	-5.09	-13.97	-18.41
1.24	-9.77	-4.40	-5.37	-13.94	-18.38
1.25	-10.02	-4.37	-5.65	-13.92	-18.35
1.26	-10.29	-4.34	-5.95	-13.89	-18.32
1.27	-10.57	-4.32	-6.25	-13.86	-18.30
1.28	-10.85	-4.29	-6.56	-13.83	-18.27
1.29	-11.15	-4.27	-6.88	-13.81	-18.24
1.30	-11.45	-4.24	-7.21	-13.78	-18.22
1.31	-11.77	-4.22	-7.56	-13.76	-18.20
1.32	-12.10	-4.19	-7.91	-13.74	-18.17
1.33	-12.45	-4.17	-8.28	-13.71	-18.15
1.34	-12.81	-4.15	-8.66	-13.69	-18.13
1.35	-13.19	-4.13	-9.06	-13.67	-18.11
1.36	-13.59	-4.11	-9.47	-13.65	-18.09
1.37	-14.00	-4.09	-9.91	-13.64	-18.07
1.38	-14.44	-4.08	-10.36	-13.62	-18.06
1.39	-14.90	-4.06	-10.84	-13.60	-18.04
1.40	-15.39	-4.05	-11.35	-13.59	-18.02
1.41	-15.91	-4.03	-11.88	-13.57	-18.01
1.42	-16.47	-4.02	-12.45	-13.56	-18.00
1.43	-17.06	-4.00	-13.05	-13.55	-17.98
1.44	-17.69	-3.99	-13.70	-13.53	-17.97
1.45	-18.38	-3.98	-14.40	-13.52	-17.96
1.46	-19.13	-3.97	-15.16	-13.51	-17.95
1.47	-19.95	-3.96	-15.98	-13.50	-17.94
1.48	-20.85	-3.95	-16.90	-13.50	-17.93
1.49	-21.86	-3.95	-17.92	-13.49	-17.93
1.50	-23.01	-3.94	-19.07	-13.48	-17.92
1.51	-24.33	-3.93	-20.39	-13.48	-17.91
1.52	-25.89	-3.93	-21.96	-13.47	-17.91
1.53	-27.79	-3.93	-23.86	-13.47	-17.90

Table 3 (continued)
Square-Wave Phase Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C3 (DB)	C5 (DB)
1.54	-30.23	-3.92	-26.31	-13.46	-17.90
1.55	-33.64	-3.92	-29.72	-13.46	-17.90
1.56	-39.33	-3.92	-35.42	-13.46	-17.90
1.57		-3.92	-58.06	-13.46	-17.90
1.58	-40.72	-3.92	-36.80	-13.46	-17.90
1.59	-34.33	-3.92	-30.41	-13.46	-17.90
1.60	-30.69	-3.92	-26.77	-13.46	-17.90
1.61	-28.14	-3.92	-24.21	-13.47	-17.90
1.62	-26.16	-3.93	-22.24	-13.47	-17.91
1.63	-24.56	-3.93	-20.62	-13.48	-17.91
1.64	-23.20	-3.94	-19.27	-13.48	-17.92
1.65	-22.03	-3.95	-18.09	-13.49	-17.92
1.66	-21.00	-3.95	-17.05	-13.50	-17.93
1.67	-20.08	-3.96	-16.12	-13.50	-17.94
1.68	-19.25	-3.97	-15.28	-13.51	-17.95
1.69	-18.49	-3.98	-14.51	-13.52	-17.96
1.70	-17.80	-3.99	-13.81	-13.53	-17.97
1.71	-17.16	-4.00	-13.15	-13.54	-17.98
1.72	-16.56	-4.02	-12.54	-13.56	-17.99
1.73	-16.00	-4.03	-11.97	-13.57	-18.01
1.74	-15.47	-4.04	-11.43	-13.59	-18.02
1.75	-14.98	-4.06	-10.92	-13.60	-18.04
1.76	-14.51	-4.07	-10.44	-13.62	-18.05
1.77	-14.07	-4.09	-9.98	-13.63	-18.07
1.78	-13.65	-4.11	-9.54	-13.65	-18.09
1.79	-13.25	-4.13	-9.12	-13.67	-18.11
1.80	-12.87	-4.15	-8.72	-13.69	-18.13
1.81	-12.51	-4.17	-8.34	-13.71	-18.15
1.82	-12.16	-4.19	-7.97	-13.73	-18.17
1.83	-11.82	-4.21	-7.61	-13.76	-18.19
1.84	-11.50	-4.24	-7.27	-13.78	-18.22
1.85	-11.19	-4.26	-6.93	-13.80	-18.24
1.86	-10.90	-4.29	-6.61	-13.83	-18.27
1.87	-10.61	-4.31	-6.30	-13.86	-18.29
1.88	-10.33	-4.34	-5.99	-13.88	-18.32
1.89	-10.07	-4.37	-5.70	-13.91	-18.35
1.90	-9.81	-4.40	-5.41	-13.94	-18.38
1.91	-9.56	-4.43	-5.13	-13.97	-18.41
1.92	-9.32	-4.46	-4.86	-14.00	-18.44
1.93	-9.08	-4.49	-4.59	-14.03	-18.47
1.94	-8.85	-4.52	-4.33	-14.07	-18.50
1.95	-8.63	-4.56	-4.07	-14.10	-18.54
1.96	-8.42	-4.59	-3.82	-14.14	-18.57
1.97	-8.21	-4.63	-3.58	-14.17	-18.61
1.98	-8.00	-4.67	-3.34	-14.21	-18.65
1.99	-7.81	-4.70	-3.10	-14.25	-18.68
2.00	-7.62	-4.74	-2.87	-14.29	-18.72
2.01	-7.43	-4.78	-2.64	-14.33	-18.76
2.02	-7.25	-4.83	-2.42	-14.37	-18.80
2.03	-7.07	-4.87	-2.20	-14.41	-18.85
2.04	-6.89	-4.91	-1.98	-14.45	-18.89
2.05	-6.72	-4.96	-1.77	-14.50	-18.94
2.06	-6.56	-5.00	-1.56	-14.54	-18.98
2.07	-6.40	-5.05	-1.35	-14.59	-19.03
2.08	-6.24	-5.10	-1.14	-14.64	-19.08
2.09	-6.09	-5.15	-0.94	-14.69	-19.12
2.10	-5.94	-5.20	-0.74	-14.74	-19.18
2.11	-5.79	-5.25	-0.54	-14.79	-19.23
2.12	-5.65	-5.30	-0.35	-14.84	-19.28
2.13	-5.51	-5.35	-0.15	-14.90	-19.33

Table 3 (continued)
Square-Wave Phase Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C3 (DB)	C5 (DB)
2.14	-5.37	-5.41	0.04	-14.95	-19.39
2.15	-5.23	-5.46	0.23	-15.01	-19.44
2.16	-5.10	-5.52	0.42	-15.06	-19.50
2.17	-4.97	-5.58	0.61	-15.12	-19.56
2.18	-4.85	-5.64	0.79	-15.18	-19.62
2.19	-4.73	-5.70	0.98	-15.24	-19.68
2.20	-4.61	-5.76	1.16	-15.31	-19.74
2.21	-4.49	-5.83	1.34	-15.37	-19.81
2.22	-4.37	-5.89	1.52	-15.44	-19.87
2.23	-4.26	-5.96	1.70	-15.50	-19.94
2.24	-4.15	-6.03	1.88	-15.57	-20.01
2.25	-4.04	-6.10	2.06	-15.64	-20.08
2.26	-3.93	-6.17	2.24	-15.71	-20.15
2.27	-3.83	-6.24	2.41	-15.78	-20.22
2.28	-3.73	-6.31	2.59	-15.86	-20.29
2.29	-3.63	-6.39	2.76	-15.93	-20.37
2.30	-3.53	-6.47	2.94	-16.01	-20.45
2.31	-3.43	-6.55	3.11	-16.09	-20.52
2.32	-3.34	-6.63	3.29	-16.17	-20.60
2.33	-3.24	-6.71	3.46	-16.25	-20.69
2.34	-3.15	-6.79	3.64	-16.33	-20.77
2.35	-3.06	-6.87	3.81	-16.42	-20.85
2.36	-2.98	-6.96	3.98	-16.50	-20.94
2.37	-2.89	-7.05	4.16	-16.59	-21.03
2.38	-2.81	-7.14	4.33	-16.68	-21.12
2.39	-2.73	-7.23	4.51	-16.77	-21.21
2.40	-2.65	-7.33	4.68	-16.87	-21.31
2.41	-2.57	-7.42	4.85	-16.96	-21.40
2.42	-2.49	-7.52	5.03	-17.06	-21.50
2.43	-2.41	-7.62	5.20	-17.16	-21.60
2.44	-2.34	-7.72	5.38	-17.26	-21.70
2.45	-2.27	-7.82	5.56	-17.37	-21.80
2.46	-2.20	-7.93	5.73	-17.47	-21.91
2.47	-2.13	-8.04	5.91	-17.58	-22.02
2.48	-2.06	-8.15	6.09	-17.69	-22.13
2.49	-1.99	-8.26	6.27	-17.80	-22.24
2.50	-1.93	-8.38	6.45	-17.92	-22.36
2.51	-1.86	-8.49	6.63	-18.04	-22.47
2.52	-1.80	-8.61	6.82	-18.16	-22.59
2.53	-1.74	-8.74	7.00	-18.28	-22.72
2.54	-1.68	-8.86	7.19	-18.40	-22.84
2.55	-1.62	-8.99	7.37	-18.53	-22.97
2.56	-1.56	-9.12	7.56	-18.66	-23.10
2.57	-1.50	-9.25	7.75	-18.80	-23.23
2.58	-1.45	-9.39	7.94	-18.93	-23.37
2.59	-1.39	-9.53	8.14	-19.07	-23.51
2.60	-1.34	-9.67	8.33	-19.22	-23.65
2.61	-1.29	-9.82	8.53	-19.36	-23.80
2.62	-1.24	-9.97	8.73	-19.51	-23.95
2.63	-1.19	-10.12	8.93	-19.66	-24.10

Table 3 (continued)
Square-Wave Phase Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C3 (DB)	C5 (DB)
2.64	-1.14	-10.28	9.14	-19.82	-24.26
2.65	-1.09	-10.44	9.34	-19.98	-24.42
2.66	-1.05	-10.60	9.55	-20.15	-24.58
2.67	-1.00	-10.77	9.77	-20.31	-24.75
2.68	-0.96	-10.94	9.98	-20.49	-24.92
2.69	-0.92	-11.12	10.20	-20.66	-25.10
2.70	-0.83	-11.30	10.43	-20.84	-25.28
2.71	-0.84	-11.49	10.65	-21.03	-25.47
2.72	-0.80	-11.68	10.88	-21.22	-25.66
2.73	-0.76	-11.88	11.12	-21.42	-25.85
2.74	-0.72	-12.06	11.36	-21.62	-26.06
2.75	-0.68	-12.2[illegible]	11.60	-21.83	-26.26
2.76	-0.65	-12.5[illegible]	11.85	-22.04	-26.48
2.77	-0.61	-12.7[illegible]	12.1[illegible]	-22.26	-26.70
2.78	-0.58	-12.94	12.36	-22.49	-26.92
2.79	-0.55	-13.18	12.63	-22.72	-27.16
2.80	-0.52	-13.42	12.90	-22.96	-27.40
2.81	-0.49	-13.67	13.18	-23.21	-27.65
2.82	-0.46	-13.92	13.46	-23.46	-27.90
2.83	-0.43	-14.19	13.75	-23.73	-28.17
2.84	-0.40	-14.46	14.06	-24.00	-28.44
2.85	-0.37	-14.75	14.37	-24.29	-28.73
2.86	-0.35	-15.04	14.69	-24.58	-29.02
2.87	-0.32	-15.35	15.02	-24.89	-29.33
2.88	-0.30	-15.66	15.36	-25.21	-29.64
2.89	-0.28	-16.00	15.72	-25.54	-29.98
2.90	-0.26	-16.34	16.08	-25.88	-30.32
2.91	-0.24	-16.70	16.47	-26.24	-30.68
2.92	-0.22	-17.08	16.86	-26.62	-31.06
2.93	-0.20	-17.47	17.28	-27.02	-31.45
2.94	-0.18	-17.89	17.71	-27.43	-31.87
2.95	-0.16	-18.32	18.16	-27.87	-32.30
2.96	-0.14	-18.78	18.64	-28.33	-32.76
2.97	-0.13	-19.27	19.14	-28.81	-33.25
2.98	-0.11	-19.79	19.67	-29.33	-33.77
2.99	-0.10	-20.34	20.24	-29.88	-34.32
3.00	-0.09	-20.93	20.84	-30.47	-34.91
3.01	-0.08	-21.56	21.48	-31.10	-35.54
3.02	-0.06	-22.24	22.18	-31.78	-36.22
3.03	-0.05	-22.98	22.93	-32.53	-36.96
3.04	-0.04	-23.80	23.75	-33.34	-37.78
3.05	-0.04	-24.69	24.66	-34.24	-38.67
3.06	-0.03	-25.69	25.67	-35.24	-39.67
3.07	-0.02	-26.83	26.81	-36.37	-40.81
3.08	-0.02	-28.13	28.12	-37.68	-42.11
3.09	-0.01	-29.67	29.66	-39.21	-43.65
3.10	-0.01	-31.54	31.53	-41.08	-45.52
3.11	-0.00	-33.93	33.92	-43.47	-47.91
3.12	-0.00	-37.23	37.23	-46.77	-51.21
3.13	-0.00	-42.63	42.63	-52.18	-56.61

SINE-WAVE FREQUENCY MODULATION AND PHASE MODULATION

Most communication theory texts contain the derivation of sideband amplitude for sinusoidal modulation. It is repeated here for convenience, and also to illustrate that in this unique case, FM and PM are essentially the same.

For the FM case, the time function can be represented as

$$f(t) = Ae^{j\phi_t}, \tag{10}$$

where

A = carrier amplitude $\equiv 1$,

$$\phi_t = \int \omega_t \, dt. \tag{11}$$

Here,

ω_t = instantaneous angular frequency,

ϕ_t = instantaneous phase of rotating vector.

Now, letting the instantaneous angular frequency be modulated by a sinusoid, we have

$$\omega_t = \omega_c + \Delta\omega \cos \omega_1 t, \tag{12}$$

where

ω_c = angular carrier frequency,

ω_1 = angular frequency of modulating signal,

$\Delta\omega$ = peak angular frequency excursion from ω_c due to modulation.

Applying this result to Equation (11) yields

$$\phi_t = \int_0^t (\omega_c + \Delta\omega \cos \omega_1 t) \, dt. \tag{13}$$

Integrating Equation (13) yields

$$\phi_t = \omega_c t + \frac{\Delta\omega}{\omega_1} \sin \omega_1 t. \tag{14}$$

Substituting Equation (14) into Equation (10) yields

$$f(t) = Ae^{j(\omega_c t + \Delta\omega/\omega_1 \sin \omega_1 t)}. \tag{15}$$

Defining $\Delta\omega/\omega_1 \equiv \beta$ in Equation (15) we have

$$f(t) = Ae^{j\omega_c t} e^{j\beta \sin \omega_1 t}. \tag{16}$$

Expanding the modulation term gives

$$e^{j\beta \sin \omega_1 t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_1 t}, \tag{17}$$

where

$$C_n = \frac{\omega_1}{2\pi} \int_{-\pi/\omega_1}^{\pi/\omega_1} e^{j\beta \sin \omega_1 t} e^{-jn\omega_1 t} dt. \tag{18}$$

Letting

$$\theta = \omega_1 t, \tag{19}$$

yields

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin\theta - n\theta)} d\theta. \tag{20}$$

Using the identity

$$e^{jx \sin \omega_1 t} = \sum_{-\infty}^{\infty} J_n(x) e^{jn\omega_1 t}, \tag{21}$$

Equation (15) takes the form

$$f(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j(\omega_c + n\omega_1)t}. \tag{22}$$

Therefore, the amplitude of the resulting FM wave is represented by the summation of Bessel terms.

For the PM case, the time function can be represented as

$$p(t) = Ae^{j\phi_t}, \tag{23}$$

where

$$\phi_t = \int \omega_t \, dt, \tag{24}$$

and

ω_t = instantaneous angular frequency

ϕ_t = instantaneous phase frequency.

Let the modulating signal change the phase such that

$$\phi_t = \phi_0 + \Delta\phi \sin \omega_1 t, \tag{25}$$

where $\phi_0 = \omega_c t$, or the phase of the unmodulated carrier.

Then,

$$p(t) = Ae^{j(\omega_c t + \Delta\phi \sin \omega_1 t)} \tag{26}$$

$$p(t) = Ae^{j\Delta\phi \sin \omega_1 t} e^{j\omega_c t}. \tag{27}$$

Notice that if we let $\Delta\phi = \beta$, then Equation (27) and Equation (16) become identical; that is, if the analysis is continued,

$$p(t) = A \sum_{n=-\infty}^{\infty} J_n(\Delta\phi) \, e^{j(\omega_c + n\omega_1)t}. \tag{28}$$

For FM, the argument of the Bessel function is the modulation index; for PM, the argument becomes the peak phase deviation. In summary, then, the resultant spectra are identical for sinusoidal modulation. Table 4 gives the sideband amplitudes as functions of modulation index. Figure 4 contains the spectral pictures.

Table 4

Sine-Wave Frequency Modulation and Phase Modulation: Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)
0.10	-0.02	-26.03	26.01	-58.0[illegible]	[illegible]3.63	-[illegible]31.[illegible]9	-171.69
0.20	-0.09	-20.04	19.96	-46.0[illegible]	-75	[illegible]107.62	-141.60
0.30	-0.20	-16.58	16.38	-39.0[illegible]	-65	[illegible]93.56	-124.01
0.40	-0.35	-14.15	13.80	-34.1[illegible]	-57	[illegible]83.59	-111.54
0.50	-0.55	-12.31	11.76	-30.28	-51.[illegible]	[illegible].88	-101.88
0.60	-0.80	-10.85	10.05	-27.20	-47.[illegible]	-69.59	-94.00
0.70	-1.10	-9.66	8.56	-24.61	-43.19	-64.29	-87.[illegible]5
0.80	-1.45	-8.66	7.21	-22.40	-39.79	-59.72	-81.61
0.90	-1.86	-7.83	5.97	-20.48	-36.81	-55.70	-76.56
1.00	-2.32	-7.13	4.81	-18.79	-34.17	-52.12	-72.05
1.10	-2.86	-6.54	3.68	-17.29	-31.80	-48.90	-67.99
1.20	-3.46	-6.05	2.59	-15.95	-29.66	-45.96	-64.29
1.30	-4.15	-5.65	1.50	-14.75	-27.72	-43.31	-60.91
1.40	-4.93	-5.32	0.39	-13.67	-25.93	-40.85	-57.79
1.50	-5.82	-5.07	-0.75	-12.69	-24.30	-38.59	-54.90
1.60	-6.83	-4.88	-1.95	-11.80	-22.79	-36.48	-52.21
1.70	-8.00	-4.76	-3.24	-11.00	-21.40	-34.52	-49.70
1.80	-9.37	-4.71	-4.66	-10.28	-20.10	-32.69	-47.34
1.90	-11.00	-4.71	-6.29	-9.63	-18.91	-30.98	-45.13
2.00	-13.00	-4.78	-8.22	-9.05	-17.79	-29.37	-43.05
2.10	-15.57	-4.91	-10.66	-8.53	-16.76	-27.86	-41.08
2.20	-19.14	-5.10	-14.04	-8.07	-15.79	-26.44	-39.22
2.30	-25.11	-5.35	-19.75	-7.66	-14.90	-25.10	-37.46
2.40	-52.01	-5.68	-46.34	-7.31	-14.06	-23.83	-35.79
2.50	-26.31	-6.07	-20.23	-7.01	-13.29	-22.64	-34.20
2.60	-20.28	-6.54	-13.74	-6.76	-12.57	-21.51	-32.69
2.70	-16.93	-7.10	-9.83	-6.57	-11.90	-20.45	-31.25
2.80	-14.65	-7.75	-6.90	-6.42	-11.29	-19.44	-29.88
2.90	-12.98	-8.51	-4.47	-6.32	-10.72	-18.49	-28.[illegible]
3.00	-11.70	-9.39	-2.30	-6.27	-10.20	-17.59	-27.32
3.10	-10.69	-10.43	-0.26	-6.26	-9.72	-16.74	-26.14
3.20	-9.89	-11.66	1.76	-6.31	-9.29	-15.93	-25.00
3.30	-9.26	-13.13	3.86	-6.41	-8.90	-15.18	-23.91
3.40	-8.77	-14.93	6.16	-6.56	-8.56	-14.46	-22.88
3.50	-8.40	-17.24	8.84	-6.77	-8.25	-13.79	-21.89
3.60	-8.14	-20.40	12.26	-7.04	-7.99	-13.16	-20.95
3.70	-7.98	-25.38	17.40	-7.36	-7.76	-12.57	-20.04
3.80	-7.90	-37.84	29.[illegible]	-7.76	-7.58	-12.02	-19.18
3.90	-7.92	-31.29	23.38	-8.23	-7.43	-11.50	-18.36
4.00	-8.02	-23.60	15.58	-8.77	-7.33	-11.02	-17.58
4.10	-8.21	-19.72	11.51	-9.41	-7.26	-10.58	-16.84
4.20	-8.48	-17.16	8.68	-10.16	-7.24	-10.17	-16.13
4.30	-8.85	-15.29	6.45	-11.02	-7.26	-9.80	-15.46
4.40	-9.31	-13.86	4.55	-12.04	-7.33	-9.46	-14.82

Table 4 (continued)
Sine-Wave Frequency Modulation and Phase Modulation:
Modulation Index VS Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C0/C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)
4.50	-9.88	-12.73	2.84	-13.24	-7.44	-9.16	-14.21
4.60	-10.57	-11.82	1.25	-14.68	-7.60	-8.89	-13.64
4.70	-11.39	-11.09	-0.31	-16.45	-7.80	-8.65	-13.10
4.80	-12.38	-10.50	-1.88	-18.71	-8.06	-8.45	-12.59
4.90	-13.57	-10.04	-3.52	-21.80	-8.38	-8.28	-12.11
5.00	-15.01	-9.69	-5.32	-26.64	-8.76	-8.15	-11.66
5.10	-16.81	-9.44	-7.37	-38.32	-9.20	-8.05	-11.24
5.20	-19.15	-9.29	-9.86	-33.26	-9.72	-7.99	-10.86
5.30	-22.41	-9.22	-13.19	-25.23	-10.32	-7.97	-10.50
5.40	-27.70	-9.23	-18.47	-21.24	-11.02	-7.98	-10.17
5.50	-43.29	-9.33	-33.96	-18.61	-11.83	-8.03	-9.87
5.60	-31.38	-9.52	-21.87	-16.69	-12.77	-8.12	-9.60
5.70	-24.45	-9.79	-14.66	-15.21	-13.88	-8.26	-9.36
5.80	-20.75	-10.14	-10.61	-14.02	-15.20	-8.43	-9.15
5.90	-18.27	-10.60	-7.67	-13.07	-16.80	-8.66	-8.97
6.00	-16.44	-11.16	-5.28	-12.29	-18.80	-8.93	-8.82
6.10	-15.03	-11.84	-3.19	-11.66	-21.45	-9.26	-8.71
6.20	-13.90	-12.66	-1.25	-11.15	-25.31	-9.65	-8.62
6.30	-13.00	-13.64	0.63	-10.76	-32.38	-10.10	-8.56
6.40	-12.28	-14.82	2.54	-10.46	-44.57	-10.62	-8.54
6.50	-11.70	-16.26	4.56	-10.25	-29.03	-11.22	-8.55
6.60	-11.24	-18.06	6.82	-10.12	-23.87	-11.91	-8.60
6.70	-10.90	-20.41	9.51	-10.07	-20.74	-12.72	-8.68
6.80	-10.66	-23.71	13.05	-10.11	-18.53	-13.65	-8.80
6.90	-10.51	-29.14	18.63	-10.22	-16.85	-14.74	-8.97
7.00	-10.46	-46.59	36.14	-10.42	-15.52	-16.04	-9.17
7.10	-10.49	-31.99	21.50	-10.69	-14.44	-17.61	-9.42
7.20	-10.60	-25.30	14.70	-11.06	-13.56	-19.57	-9.72
7.30	-10.81	-21.66	10.86	-11.52	-12.84	-22.15	-10.07
7.40	-11.10	-19.20	8.10	-12.08	-12.24	-25.85	-10.48
7.50	-11.49	-17.38	5.89	-12.76	-11.77	-32.46	-10.95
7.60	-11.99	-15.96	3.97	-13.57	-11.39	-50.10	-11.49
7.70	-12.59	-14.83	2.24	-14.54	-11.10	-30.54	-12.12
7.80	-13.33	-13.92	0.59	-15.71	-10.89	-25.08	-12.83
7.90	-14.23	-13.18	-1.04	-17.15	-10.77	-21.83	-13.66
8.00	-15.31	-12.59	-2.72	-18.94	-10.72	-19.55	-14.62
8.10	-16.62	-12.12	-4.50	-21.27	-10.74	-17.81	-15.74
8.20	-18.26	-11.77	-6.49	-24.54	-10.84	-16.44	-17.08
8.30	-20.35	-11.51	-8.84	-29.90	-11.02	-15.33	-18.71
8.40	-23.20	-11.35	-11.86	-46.59	-11.28	-14.41	-20.75
8.50	-27.55	-11.27	-16.27	-33.02	-11.61	-13.65	-23.46
8.60	-36.70	-11.28	-25.42	[illegible]	-12.04	-13.02	-27.46
8.70	-38.05	-11.38	-26.66	-22.55	-12.56	-12.51	-35.09
8.80	-28.13	-11.57	-16.56	-20.07	-13.19	-12.09	-43.11
8.90	-23.71	-11.84	-11.87	-18.22	-13.95	-11.76	-30.10
9.00	-20.88	-12.21	-8.68	[illegible]	-14.85	-11.52	-25.19
9.10	-18.84	-12.67	-6.17	[illegible]	-15.93	-11.35	-22.14
9.20	-17.28	-13.25	-4.03	-14.70	-17.24	-11.26	-19.95
9.30	-16.05	-13.96	-2.08	-13.95	-18.86	-11.23	-18.28
9.40	-15.05	-14.82	-0.24	-13.33	-20.92	-11.28	-16.94
9.50	-14.25	-15.85	1.60	-12.85	-23.70	-11.40	-15.85
9.60	-13.60	-17.11	3.51	-12.47	-27.89	-11.59	-14.94
9.70	-13.08	-18.66	5.58	-12.19	-36.33	-11.86	-14.19
9.80	-12.68	-20.65	7.97	-12.00	-40.26	-12.21	-13.56
9.90	-12.38	-23.30	10.92	[illegible]	-29.29	-12.64	-13.04
10.00	-12.18	-27.24	15.05	-11.88	-24.67	-13.17	-12.61

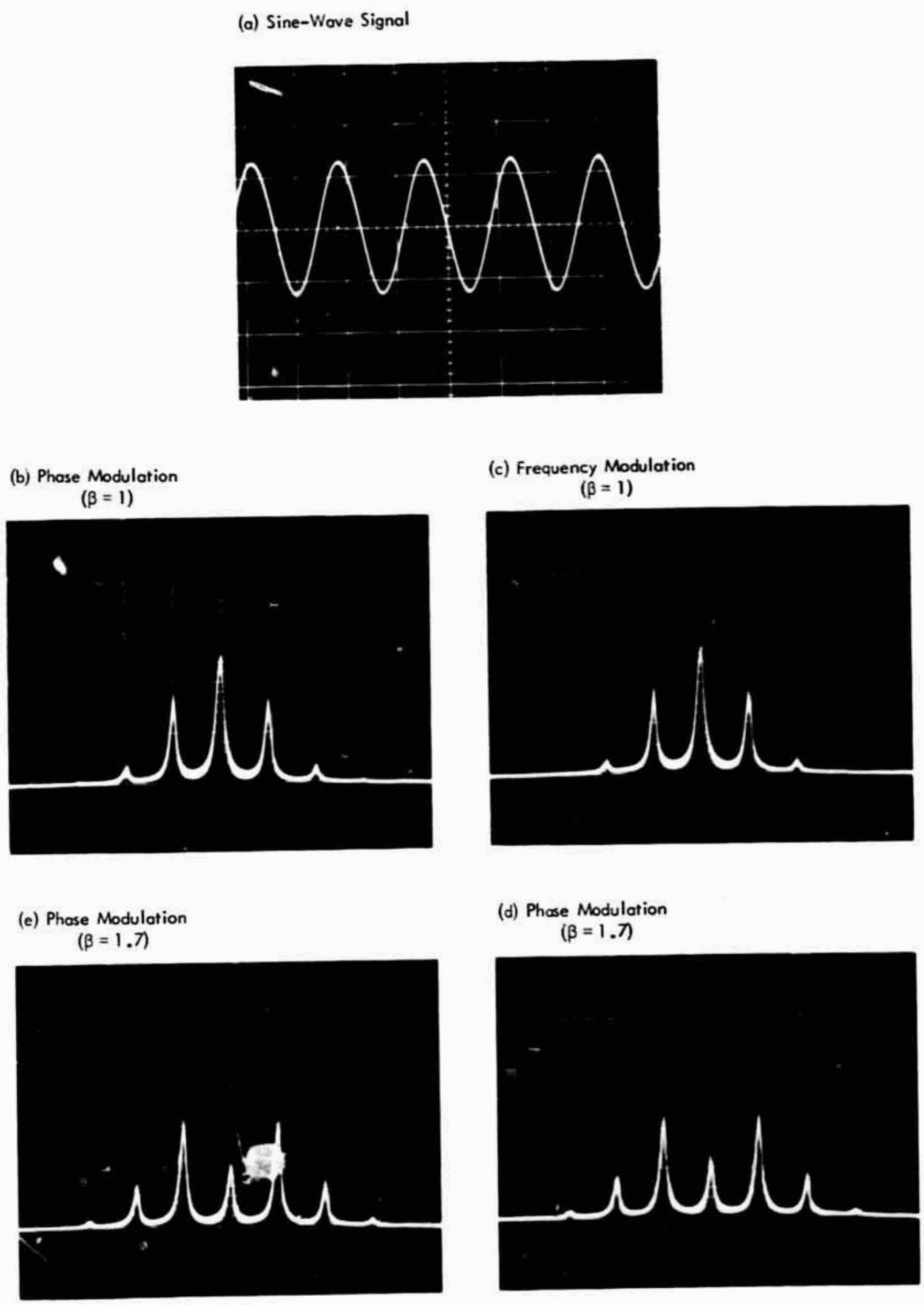


Figure 4. Frequency and Phase Modulation Spectra (Sine Wave)

ACKNOWLEDGEMENTS

The author wishes to thank Mr. Charles Scaffidi and Mr. John B. Martin for providing encouragement, reviewing this document and making many signficant suggestions; he is also grateful to Mr. Earl Turner for providing assistance in the mathematical analyses.

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APPENDIX A

ANALYSIS OF SQUARE-WAVE FREQUENCY MODULATION AND PHASE MODULATION

The following analysis develops equations for the amplitudes of sidebands resulting from the square-wave frequency modulation or phase modulation process. The generalized expression (Reference 1, Equation 18-57) for sideband amplitude is

$$C_n = \frac{1}{T_m} \int_{-T_m/2}^{T_m/2} e^{j\phi(t)} e^{-jn\omega_m t} \, dt, \tag{A1}$$

where

C_n = amplitude of n^{th} sideband,

T_m = period of modulating signal,

$\phi(t)$ = instantaneous phase of modulated signal,

ω_m = radian frequency of modulating signal,

n = sideband order (0, 1, 2, . . .).

SQUARE-WAVE PHASE MODULATION

The phase modulating function is shown in Figure A1, with phase deviations of ±B radians about the nominal phase of the unmodulated carrier. Also, $T_m = 1/f_m$, and $\omega_m = 2\pi f_m$.

To solve for the amplitude of the residual carrier, set n = 0 in Equation (A1). This yields

$$C_0 = f_m \int_{-T_m/2}^{T_m/2} e^{j\phi(t)} \, dt, \tag{A2}$$

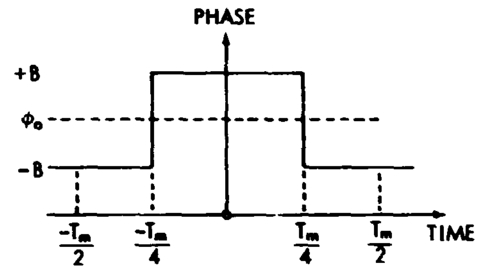


Figure A1. Square-Wave Phase Modulating Function

$$C_0 = f_m \int_{-T_m/2}^{-T_m/4} e^{-jB} dt + f_m \int_{-T_m/4}^{T_m/4} e^{jB} dt + f_m \int_{T_m/4}^{T_m/2} e^{-jB} dt,$$

$$C_0 = f_m e^{-jB} \left[\frac{-T_m}{4} + \frac{T_m}{2}\right] + f_m e^{jB} \left[\frac{T_m}{4} + \frac{T_m}{4}\right] + f_m e^{-jB} \left[\frac{T_m}{2} - \frac{T_m}{4}\right], \qquad \text{(A3)}$$

$$C_0 = f_m e^{-jB} \frac{T_m}{2} + f_m e^{jB} \frac{T_m}{2}, \qquad \text{(A4)}$$

combining terms

$$C_0 = \frac{1}{2}(e^{jB} + e^{-jB}). \qquad \text{(A5)}$$

Using $\cos x = (e^{jx} + e^{-jx})/2$, the absolute magnitude of the residual carrier amplitude becomes

$$|C_0| = |\cos B|. \qquad \text{(A6)}$$

The first sideband amplitude is obtained by setting n = 1 in equation (A1). Therefore,

$$C_1 = f_m \int_{-T_m/2}^{T_m/2} e^{j\phi(t)} e^{-j\omega_m t} dt. \qquad \text{(A7)}$$

Substituting the proper limits, we have

$$C_1 = f_m \int_{-T_m/2}^{-T_m/4} e^{-jB} e^{-j\omega_m t}\, dt$$

$$+ f_m \int_{-T_m/4}^{T_m/4} e^{jB} e^{-j\omega_m t}\, dt + f_m \int_{T_m/4}^{T_m/2} e^{-jB} e^{-j\omega_m t}\, dt, \tag{A8}$$

$$C_1 = \left(\frac{f_m e^{-jB}}{-j\omega_m}\right) e^{-j\omega_m t} \Bigg|_{-T_m/2}^{-T_m/4} + \left(\frac{f_m e^{jB}}{-j\omega_m}\right) e^{-j\omega_m t} \Bigg|_{-T_m/4}^{T_m/4} + \left(\frac{f_m e^{-jB}}{-j\omega_m}\right) e^{-j\omega_m t} \Bigg|_{T_m/4}^{T_m/2}, \tag{A9}$$

$$C_1 = \frac{e^{-jB}}{-j2\pi}\left[e^{j\omega_m T_m/4} - e^{j\omega_m T_m/2}\right] + \frac{e^{jB}}{-j2\pi}\left[e^{-j\omega_m T_m/4} - e^{j\omega_m T_m/4}\right]$$

$$+ \frac{e^{-jB}}{-j2\pi}\left[e^{-j\omega_m T_m/2} - e^{-j\omega_m T_m/4}\right], \tag{A10}$$

$$C_1 = \frac{e^{-jB}}{-j2\pi}\left[e^{j\omega_m T_m/4} - e^{-j\omega_m T_m/4}\right] + \frac{e^{-jB}}{-j2\pi}\left[-e^{j\omega_m T_m/2} + e^{-j\omega_m T_m/2}\right]$$

$$+ \frac{e^{jB}}{-j2\pi}\left[-e^{j\omega_m T_m/4} + e^{-j\omega_m T_m/4}\right], \tag{A11}$$

combining terms

$$C_1 = \frac{e^{-jB}}{-j2\pi}\left[e^{j\pi/2} - e^{-j\pi/2}\right] + \frac{e^{-jB}}{j2\pi}\left[e^{j\pi} + e^{-j\pi}\right]$$

$$+ \frac{e^{jB}}{j2\pi}\left[e^{j\pi/2} - e^{-j\pi/2}\right]. \tag{A12}$$

Using

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j},$$

Equation (A12) becomes

$$C_1 = -\frac{e^{-jB}}{\pi}\sin\frac{\pi}{2} + \frac{e^{-jB}}{\pi}\sin\pi + \frac{e^{jB}}{\pi}\sin\frac{\pi}{2}.$$

$$C_1 = \frac{1}{\pi}\left(e^{jB} - e^{-jB}\right), \quad C_1 = \frac{2j}{\pi}\sin B. \tag{A13}$$

Taking the absolute magnitude of Equation (A13) yields

$$|C_1| = \left|\frac{2}{\pi}\sin B\right|. \tag{A14}$$

Therefore, the ratio of residual carrier level to first sideband amplitude becomes

$$\left|\frac{C_0}{C_1}\right| = \left|\frac{\pi}{2}\cot B.\right| \tag{A15}$$

The second-order sideband is

$$C_2 = f_m \int_{-T_m/2}^{T_m/2} e^{j\phi(t)}\, e^{-2j\omega_m t}\, dt. \tag{A16}$$

Substituting the limits yields

$$C_2 = f_m \int_{-T_m/2}^{-T_m/4} e^{-jB} e^{-2j\omega_m t}\, dt + f_m \int_{-T_m/4}^{T_m/4} e^{jB} e^{-2j\omega_m t}\, dt$$

$$+ f_m \int_{T_m/4}^{T_m/2} e^{-jB}\, e^{-2j\omega_m t} dt, \tag{A17}$$

$$C_2 = \frac{f_m e^{-jB}}{-2j\omega_m}\left[e^{2j\omega_m T_m/4} - e^{2j\omega_m T_m/2}\right] + \frac{f_m e^{jB}}{-2j\omega_m}\left[e^{-2j\omega_m T_m/4}\right.$$

$$\left. - e^{2j\omega_m T_m/4}\right] + \frac{f_m e^{-jB}}{-2j\omega_m}\left[e^{-2j\omega_m T_m/2} - e^{-2j\omega_m T_m/4}\right], \tag{A18}$$

$$C_2 = \frac{e^{-jB}}{-4\pi j}[e^{j\pi} - e^{-j\pi}] + \frac{e^{-jB}}{4\pi j}[e^{j2\pi} - e^{-j2\pi}] + \frac{e^{jB}}{4\pi j}[e^{j\pi} - e^{-j\pi}], \qquad \text{(A19)}$$

combining terms

$$C_2 = \frac{e^{-jB}}{-2\pi}\sin\pi + \frac{e^{-jB}}{2\pi}\sin 2\pi + \frac{e^{jB}}{2\pi}\sin\pi.$$

Therefore, $C_2 = 0$. All higher-order even terms in multiples of π and 2π in the above expression would cause all even terms to be zero. Solving for the third sideband amplitude gives

$$C_3 = f_m \int_{-T_m/2}^{T_m/2} e^{j\phi(t)}\, e^{-3j\omega_m t}\, dt, \qquad \text{(A20)}$$

$$C_3 = \frac{f_m e^{-jB}}{-3j\omega_m}\left[e^{3j\omega_m T_m/4} - e^{3j\omega_m T_m/2}\right] + \frac{f_m e^{jB}}{-3j\omega_m}\left[e^{-3j\omega_m T_m/4} - e^{3j\omega_m T_m/4}\right]$$

$$+ \frac{f_m e^{-jB}}{-3j\omega_m}\left[e^{-3j\omega_m T_m/2} - e^{-3j\omega_m T_m/4}\right],$$

$$C_3 = \frac{e^{-jB}}{-6\pi j}[e^{j\,3/2\,\pi} - e^{-j\,3/2\,\pi}] + \frac{e^{-jB}}{6\pi j}[e^{j3\pi} - e^{-j3\pi}]$$

$$+ \frac{e^{jB}}{6\pi j}[e^{j\,3/2\,\pi} - e^{-j\,3/2\,\pi}], \qquad \text{(A24)}$$

$$C_3 = \frac{e^{-jB}}{-3\pi}\sin\frac{3}{2}\pi + \frac{e^{-jB}}{3\pi}\sin 3\pi + \frac{e^{jB}}{3\pi}\sin\frac{3}{2}\pi, \qquad \text{(A25)}$$

$$C_3 = \frac{2j}{3\pi}\sin B.$$

Solving for the fifth- and seventh-order sideband would yield the expressions:

$$C_5 = \frac{2j}{5\pi}\sin B \qquad \text{(A26)}$$

$$C_7 = \frac{2j}{7\pi} \sin B. \tag{A27}$$

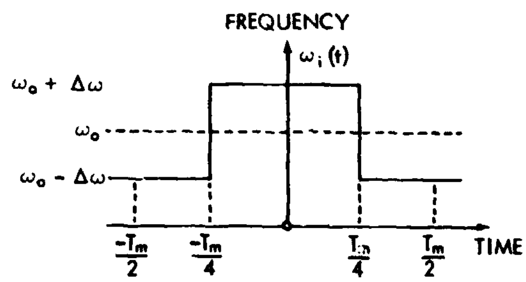
In general, then,

$$|C_n| = \left|\frac{2}{n\pi} \sin B\right|. \quad (n = 1, 3, 5, \ldots). \tag{A28}$$

SQUARE-WAVE FREQUENCY MODULATION

The frequency-modulating function is shown in Figure A2 with frequency deviations of $\pm \Delta\omega$ rad/sec about the normal frequency of the unmodulated carrier.

Figure A2. Square-Wave Frequency Modulating Function



The instantaneous frequency is given by

$$\begin{aligned} \omega_i(t) &= \omega_c - \Delta\omega; \left(-\frac{T_m}{2} < t < -\frac{T_m}{4}\right) \\ &= \omega_c + \Delta\omega; \left(-\frac{T_m}{4} < t < \frac{T_m}{4}\right) \\ &= \omega_c - \Delta\omega; \left(\frac{T_m}{4} < t < \frac{T_m}{2}\right). \end{aligned} \tag{A29}$$

The instantaneous phase is given by

$$\begin{aligned} \phi_i(t) &= \omega_c t - \frac{\Delta\omega}{\omega_m}(\pi + \omega_m t); \left(\frac{-\pi}{\omega_m} < t < \frac{-\pi}{2\omega_m}\right) \\ &= \omega_c t + \Delta\omega t; \left(\frac{-\pi}{2\omega_m} < t < \frac{\pi}{2\omega_m}\right) \\ &= \omega_c t + \frac{\Delta\omega}{\omega_m}(\pi - \omega_m t); \left(\frac{\pi}{2\omega_m} < t < \frac{\pi}{\omega_m}\right). \end{aligned} \tag{A30}$$

The phase variations about the nominal carrier phase are given by

$$\phi(t) = -\frac{\Delta\omega}{\omega_m}(\pi + \omega_m t); \left(-\frac{\pi}{\omega_m} < t < -\frac{\pi}{2\omega_m}\right)$$

$$= \Delta\omega t; \left(-\frac{\pi}{2\omega_m} < t < \frac{\pi}{2\omega_m}\right)$$

$$= \frac{\Delta\omega}{\omega_m}(\pi - \omega_m t); \left(\frac{\pi}{2\omega_m} < t < \frac{\pi}{\omega_m}\right). \quad \text{(A31)}$$

Substituting these values into Equation (A1), the following general expression results:

$$C_n = f_m \int_{-\pi/\omega_m}^{-\pi/2\omega_m} e^{-j\beta\pi} e^{-j(\beta\omega_m t + n\omega_m t)} dt$$

$$+ f_m \int_{-\pi/2\omega_m}^{\pi/2\omega_m} e^{j\beta\omega_m t} e^{-jn\omega_m t} dt$$

$$+ f_m \int_{\pi/2\omega_m}^{\pi/\omega_m} e^{j\beta\pi} e^{-j(\beta\omega_m t + n\omega_m t)}, \quad \text{(A32)}$$

where

$$\frac{\Delta\omega}{\omega_m} \equiv \beta.$$

Letting

$$\theta = \omega_m t$$

$$d\theta = \omega_m dt, \quad \text{(A33)}$$

changing the limits and setting n = 0, the carrier term becomes

$$C_0 = \frac{e^{-j\beta\pi}}{2\pi} \int_{-\pi}^{-\pi/2} e^{-j\beta\theta}\, d\theta + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\beta\theta}\, d\theta$$

$$+ \frac{e^{j\beta\pi}}{2\pi} \int_{\pi/2}^{\pi} e^{-j\beta\theta}\, d\theta, \tag{A34}$$

$$C_0 = \frac{e^{-j\beta\pi}}{-2\pi j\beta} e^{-j\beta\theta} \Bigg|_{-\pi}^{-\pi/2} + \frac{1}{2\pi j\beta} e^{j\beta\theta} \Bigg|_{-\pi/2}^{\pi/2} + \frac{e^{j\beta\pi}}{-2\pi j\beta} e^{-j\beta\theta} \Bigg|_{\pi/2}^{\pi}, \tag{A35}$$

$$C_0 = \frac{e^{-j\beta\pi}}{-2\pi j\beta} [e^{j\beta\pi/2} - e^{j\beta\pi}] + \frac{1}{2\pi j\beta} [e^{j\beta\pi/2} - e^{-j\beta\pi/2}]$$

$$+ \frac{e^{j\beta\pi}}{-2\pi j\beta} [e^{-j\beta\pi} - e^{-j\beta\pi/2}], \tag{A36}$$

$$C_0 = \frac{1}{\pi\beta} \sin\beta\frac{\pi}{2} + 1 - 1 + \frac{1}{\pi\beta} \sin\beta\frac{\pi}{2}, \tag{A37}$$

$$|C_0| = \left|\frac{2}{\pi\beta} \sin\beta\frac{\pi}{2}\right|. \tag{A38}$$

Setting n = 1, the first sideband term becomes

$$C_1 = \frac{1}{2\pi} e^{-j\beta\pi} \int_{-\pi}^{-\pi/2} e^{-j(\beta+1)\theta}\, d\theta + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j(\beta-1)\theta}\, d\theta$$

$$+ e^{j\beta\pi} \int_{\pi/2}^{\pi} e^{-j(\beta-1)\theta}\, d\theta, \tag{A39}$$

$$C_1 = \frac{e^{-j\beta\pi}}{-2\pi j(\beta+1)} e^{-j(\beta+1)\theta} \Bigg|_{-\pi}^{-\pi/2} + \frac{1}{2\pi j(\beta-1)} e^{j(\beta-1)\theta} \Bigg|_{-\pi/2}^{\pi/2}$$

$$+ \frac{e^{j\beta\pi}}{-2\pi j(\beta-1)} e^{-j(\beta-1)\theta} \Bigg|_{\pi/2}^{\pi}. \tag{A40}$$

Substituting the limits and simplifying, the final expression becomes

$$|C_1| = \left| \frac{2\beta \sin \left[(\beta - 1)\frac{\pi}{2}\right]}{\pi(\beta^2 - 1)} \right| . \tag{A41}$$

If the process is continued for higher-order values of n, the following general expression is obtained:

$$|C_n| = \left| \frac{2\beta}{\pi(\beta^2 - n^2)} \sin \left[(\beta - n)\frac{\pi}{2}\right] \right| . \tag{A42}$$

APPENDIX B

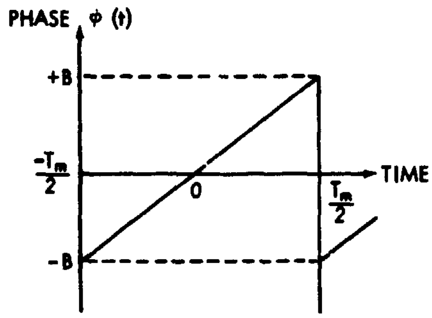
To illustrate the generality of the analysis performed in Appendix A, the PM spectrum of two additional functions is derived; specifically the sawtooth and triangular wave functions.

SAWTOOTH WAVE PHASE MODULATION

The sawtooth function is shown in Figure B1 with peak phase deviation of $\pm$ B radians. Again starting with the general expression for the n^{th} sideband amplitude,

$$C_n = \frac{1}{T_m} \int_{-T_{m/2}}^{T_{m/2}} e^{j\phi(t)} \, e^{-jn\omega_m t} \, dt. \qquad \text{(B1)}$$

Figure B1. Sawtooth-Wave Phase Modulating Function



Defining the phase function as

$$\phi(t) = \frac{2B}{T_m} t. \qquad \text{(B2)}$$

Substituting Equation (B2) into (B1) yields

$$C_n = \frac{1}{T_m} \int_{-T_{m/2}}^{T_{m/2}} e^{j\,2B/T_m\,t} \, e^{-jn\omega_m t} \, dt. \qquad \text{(B3)}$$

Setting n = 0 for the residual carrier term yields

$$C_0 = \frac{1}{T_m} \int_{-T_m/2}^{T_m/2} e^{j\,2B/T_m\ t}\ dt, \tag{B4}$$

$$C_0 = \frac{1}{j\,T_m\,\frac{2B}{T_m}}\, e^{j\,2B/T_m\ t} \Bigg|_{-T_m/2}^{T_m/2}, \tag{B5}$$

$$|C_0| = \left|\frac{\sin B}{B}\right|. \tag{B6}$$

Setting n = 1 for the first sideband amplitude yields

$$C_1 = \frac{1}{T_m} \int_{-T_m/2}^{T_m/2} e^{j\ 2B/T_m\ t}\ e^{-j\,\omega_m t}\ dt, \tag{B7}$$

$$C_1 = \frac{1}{j\,T_m\left(\frac{2B}{T_m} - \omega_m\right)}\, e^{j\,t\,(2B/T_m - \omega_m)} \Bigg|_{-T_m/2}^{T_m/2}, \tag{B8}$$

$$C_1 = \frac{2j}{j\,(2B - T_m\omega_m)}\, \sin \frac{T_m}{2}\left(\frac{2B}{T_m} - \omega_m\right). \tag{B9}$$

Using

$$\omega_m = 2\pi f_m$$

$$T_m = \frac{1}{f_m}$$

and substituting into Equation (B9), we have

$$|C_1| = \left|\frac{\sin(B - \pi)}{(B - \pi)}\right|. \tag{B10}$$

If the process is continued, the following general expression results:

$$|C_n| = \left| \frac{\sin(B - n\pi)}{(B - n\pi)} \right| , \tag{B11}$$

where n = 0, 1, 2, ...

The results of this equation are tabulated in Table B1, and the associated spectrum pictures are shown in Figure B2. The spectral asymmetry appears to be unique to this modulating function. The sideband amplitude equation is used for the higher level set – in this case, the left sideband set.

TRIANGULAR WAVE PHASE MODULATION

The triangular wave modulating function is shown in Figure B3.

Defining the function as follows

$$\phi(t) = \frac{4B}{T_m} \qquad -\frac{T_m}{2} \le t \le 0$$

$$= -\frac{4B}{T_m} t + B \ ; \quad 0 \le t \le \frac{T_m}{2} \tag{B12}$$

Again the general expression for the n^{th} sideband amplitude is given by

$$C_n = \frac{1}{T_m} \int_{-T_m/2}^{T_m/2} e^{j\phi(t)} e^{-jn\omega_m t} dt \tag{B13}$$

$$C_n = \frac{1}{T_m} \int_{-T_m/2}^{T_m/2} e^{j(\phi(t) - n\omega_m t)} dt \tag{B14}$$

$$C_n = \frac{1}{T_m} \int_{-T_m/2}^{0} e^{j(4B/T_m\, t + B - n\omega_m t)} dt$$

$$+ \frac{1}{T_m} \int_{0}^{T_m/2} e^{j(-4B/T_m\, t + B - n\omega_m t)} dt \tag{B15}$$

Table B1

Sawtooth-Wave Phase Modulation: Modulation Index VS Sideband Amplitude

BETA (RAD)	CO (DB)	C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)
.10	-.01448	-29.67650	-35.83872	-39.40725	-41.92928	-43.88140
.20	-.05798	-23.40903	-29.72000	-33.33650	-36.88222	-37.84847
.30	-.13068	-19.65949	-26.12690	-29.79270	-32.36257	-34.34316
.40	-.23287	-16.95173	-23.58392	-27.30040	-29.89489	-31.89002
.50	-.36497	-14.82289	-21.62891	-25.39752	-28.01710	-30.02699
.60	-.52755	-13.06664	-20.05636	-23.87860	-26.52377	-28.54864
.70	-.72130	-11.57281	-18.75698	-22.63443	-25.30570	-27.34578
.80	-.94710	-10.27553	-17.66596	-21.60026	-24.29815	-26.35367
.90	-1.20598	-9.13227	-16.74192	-20.73480	-23.45985	-25.53104
1.00	-1.49921	-8.11395	-15.95713	-20.01038	-22.76315	-24.85026
1.10	-1.82827	-7.19980	-15.29235	-19.40787	-22.18894	-24.29221
1.20	-2.19494	-6.37447	-14.73403	-18.91380	-21.72375	-23.84344
1.30	-2.60130	-5.62631	-14.27258	-18.51867	-21.35812	-23.49449
1.40	-3.04987	-4.94624	-13.90137	-18.21597	-21.08555	-23.23886
1.50	-3.54361	-4.32709	-13.61612	-18.00152	-20.90188	-23.07242
1.60	-4.08610	-3.76309	-13.41453	-17.87314	-20.80496	-22.99299
1.70	-4.68168	-3.24955	-13.29605	-17.83042	-20.79439	-23.00020
1.80	-5.33562	-2.78259	-13.26191	-17.87472	-20.87156	-23.09545
1.90	-6.05449	-2.35900	-13.31521	-18.00929	-21.03975	-23.28201
2.00	-6.84648	-1.97610	-13.46121	-18.23955	-21.30439	-23.56533
2.10	-7.72206	-1.63163	-13.70781	-18.57356	-21.67359	-23.95353
2.20	-8.69489	-1.32369	-14.06641	-19.02292	-22.15897	-24.45823
2.30	-9.78321	-1.05069	-14.55326	-19.60409	-22.77699	-25.09591
2.40	-11.01219	-.81127	-15.19172	-20.34061	-23.55128	-25.89019
2.50	-12.41792	-.60430	-16.01627	-21.26723	-24.51657	-26.87583
2.60	-14.05487	-.42885	-17.07987	-22.43717	-25.72614	-28.10610
2.70	-16.01099	-.28415	-18.46910	-23.93727	-27.26686	-29.66790
2.80	-18.44257	-.16958	-20.33894	-25.92283	-29.29407	-31.71656
2.90	-21.67094	-.08465	-23.00950	-28.71430	-32.12825	-34.57258
3.00	-26.55065	-.02904	-27.33413	-33.16539	-36.62317	-39.08974
3.10	-37.44940	-.00250	-37.67941	-43.64307	-47.14584	-49.63507
3.20	-34.77858	-.00493	-34.45558	-40.55806	-44.10701	-46.61932
3.30	-26.41112	-.03635	-25.53445	-31.78265	-35.37904	-37.91485
3.40	-22.48036	-.09688	-21.04823	-27.44960	-31.09473	-33.65450
3.50	-19.98058	-.18676	-17.99006	-24.55266	-28.24789	-30.83208
3.60	-18.20738	-.30636	-15.65434	-22.38692	-26.13366	-28.74275
3.70	-16.88120	-.45619	-13.76027	-20.67234	-24.47208	-27.10655
3.80	-15.86266	-.63688	-12.16717	-19.26909	-23.12338	-25.78373
3.90	-15.07247	-.84923	-10.79434	-18.09748	-22.00792	-24.69468
4.00	-14.46154	-1.09421	-9.59117	-17.10798	-21.07628	-23.78997
4.10	-13.99766	-1.37299	-8.52380	-16.26802	-20.29593	-23.03711
4.20	-13.65888	-1.68695	-7.56845	-15.55525	-19.64463	-22.41386
4.30	-13.42988	-2.03774	-6.70778	-14.95401	-19.10681	-21.90467
4.40	-13.29994	-2.42731	-5.92875	-14.45322	-18.67147	-21.49858
4.50	-13.26164	-2.85799	-5.22132	-14.04513	-18.33096	-21.18793
4.60	-13.31012	-3.33254	-4.57761	-13.72451	-18.08018	-20.96766
4.70	-13.44262	-3.85428	-3.99130	-13.48829	-17.9[illegible]15	-20.83480
4.80	-13.65520	-4.42720	-3.45728	-13.33519	-17.83774	-20.78825
4.90	-13.95768	-5.05615	-2.97137	-13.26571	-17.84556	-20.82864
5.00	-14.34371	-5.74713	-2.53010	-13.28214	-17.94206	-20.95845

Table B1 (Continued)
Sawtooth-Wave Phase Modulation: Modulation Index VS Sideband Amplitude

BETA (RAD)	CO (DB)	C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)
5.10	-14.82092	-6.50757	-2.13057	-13.38879	-18.13170	-21.18215
5.20	-15.39638	-7.34694	-1.77036	-13.59239	-18.42138	-21.50668
5.30	-16.08025	-8.27741	-1.44744	-13.90275	-18.82109	-21.94205
5.40	-16.88693	-9.31510	-1.16009	-14.33389	-19.34504	-22.50251
5.50	-17.83681	-10.48193	-.90685	-14.90586	-20.01349	-23.20834
5.60	-18.95950	-11.80882	-.68651	-15.64786	-20.85587	-24.08901
5.70	-20.29942	-13.34132	-.49805	-16.60393	-21.91647	-25.18884
5.80	-21.92693	-15.15080	-.34064	-17.84400	-23.26549	-26.57806
5.90	-23.96247	-17.35860	-.21360	-19.48806	-25.02322	-28.37700
6.00	-26.63801	-20.19747	-.11640	-21.76763	-27.42149	-30.81755
6.10	-30.49741	-24.21197	-.04863	-25.22607	-31.00403	-34.44344
6.20	-37.45692	-31.31899	-.01001	-31.77908	-37.68692	-41.17083
6.30	-51.47344	-45.47599	-.00040	-45.38301	-51.42695	-54.95654
6.40	-34.79341	-28.92992	-.01976	-28.28368	-34.47040	-38.04690
6.50	-29.60465	-23.86905	-.06815	-22.66824	-29.00496	-32.62965
6.60	-26.52056	-20.90720	-.14579	-19.14937	-25.64389	-29.31811
6.70	-24.37561	-18.87923	-.25297	-16.56074	-23.22150	-26.94665
6.80	-22.77164	-17.38930	-.39016	-14.50531	-21.34151	-25.11903
6.90	-21.53181	-16.25491	-.55793	-12.79930	-19.82093	-23.65234
7.00	-20.55083	-15.37703	-.75700	-11.34230	-18.56031	-22.44720
7.10	-19.77098	-14.69622	-.98828	-10.07344	-17.49979	-21.44382
7.20	-19.15387	-14.17433	-1.25285	-8.95295	-16.60083	-20.60373
7.30	-18.67361	-13.78570	-1.55199	-7.95343	-15.83738	-19.90097
7.40	-18.31226	-13.51257	-1.88724	-7.05517	-15.19133	-19.31751
7.50	-18.05716	-13.34249	-2.26043	-6.24355	-14.64988	-18.84065
7.60	-17.89948	-13.26680	-2.67370	-5.50741	-14.20399	-18.46144
7.70	-17.83319	-13.27964	-3.12964	-4.83804	-13.84748	-18.17382
7.80	-17.85455	-13.37741	-3.63131	-4.22854	-13.57641	-17.97395
7.90	-17.96174	-13.55843	-4.18240	-3.67333	-13.38878	-17.85995
8.00	-18.15672	-13.82280	-4.78739	-3.16787	-13.28435	-17.83172
8.10	-18.43525	-14.17239	-5.45176	-2.70841	-13.26464	-17.89092
8.20	-18.80698	-14.61097	-6.18229	-2.29184	-13.33310	-18.04113
8.30	-19.27577	-15.14452	-6.98753	-1.91555	-13.49538	-18.28819
8.40	y19.85033	-15.78183	-7.87841	-1.57734	-13.75991	-18.64068
8.50	-20.54302	-16.53535	-8.86922	-1.27539	-14.13878	-19.11089
8.60	-21.37134	-17.42269	-9.97920	-1.00812	-14.64924	-19.71628
8.70	-22.36051	-18.46913	-11.23499	-.77423	-15.31620	-20.48196
8.80	-23.54776	-19.71199	-12.67513	-.57262	-16.17657	-21.44511
8.90	y24.99068	-21.20893	-14.35834	-.40239	-17.28761	-22.66322
9.00	-26.78440	-23.05514	-16.38075	-.26279	-18.74408	-24.23137
9.10	-29.10233	-25.42411	-18.91667	-.15323	-20.71900	-26.32286
9.20	-32.31394	-28.68536	-22.33636	-.07326	-23.58143	-29.30709
9.30	-37.46945	-33.88916	-27.69076	-.02255	-28.38110	-34.23419
9.40	-51.58213	-48.04885	-41.99379	-.00088	-42.13081	-48.11736
9.50	-42.03576	-38.54825	-32.62981	-.00819	-32.21378	-38.34027
9.60	-34.81816	-31.38522	-25.58714	-.04449	-24.61722	-30.89064
9.70	-31.05170	-27.65223	-21.98869	-.10993	-20.46291	-26.89081
9.80	-28.54353	-25.18642	-19.64200	-.20478	-17.55722	-24.14777
9.90	-26.70420	-23.38838	-17.95802	-.32942	-15.30990	-22.07197
10.00	-25.24768	-22.01215	-16.69110	-.48439	-13.47407	-20.41730

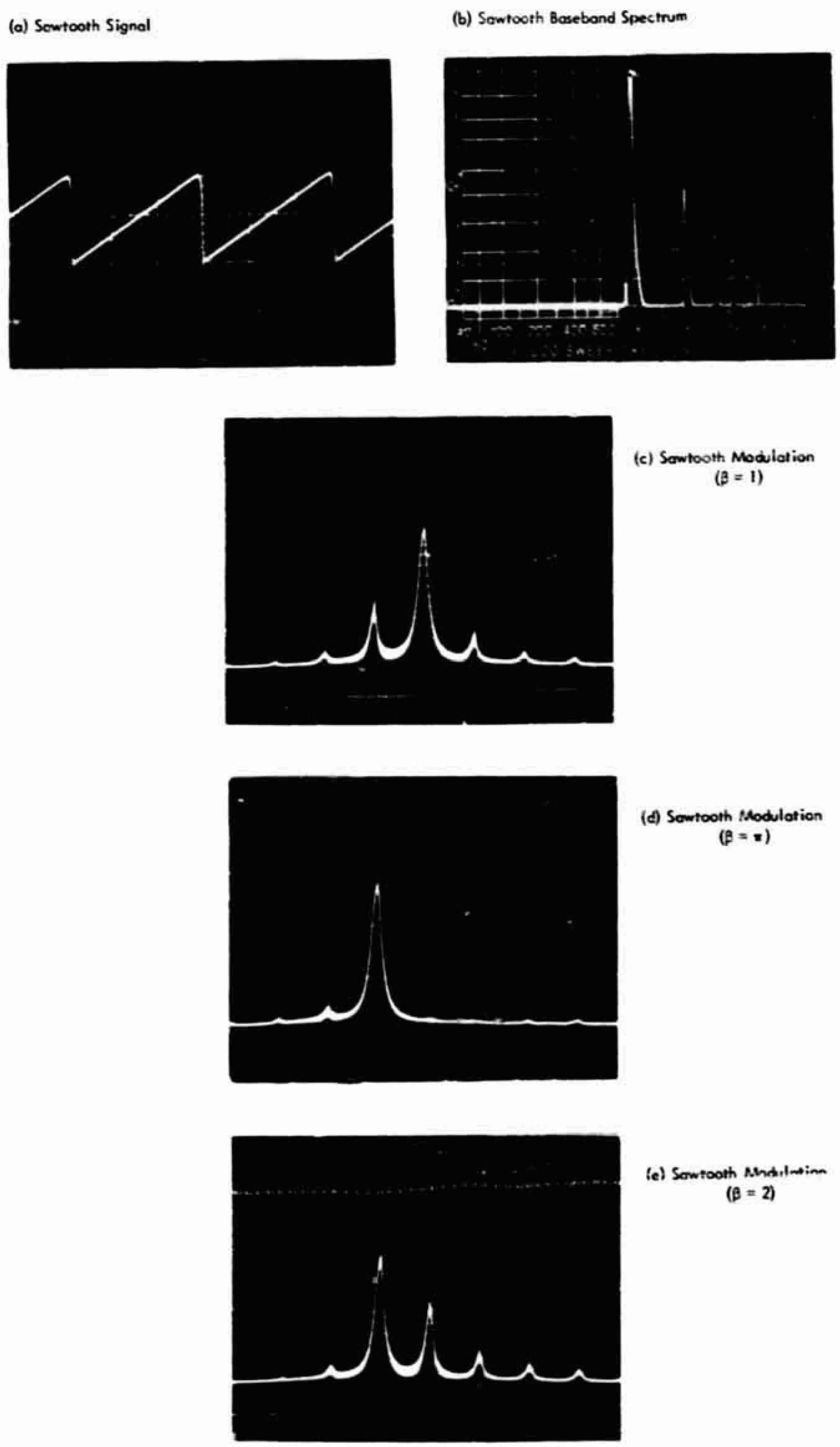


Figure B2. Phase Modulation Spectra (Sawtooth Wave)

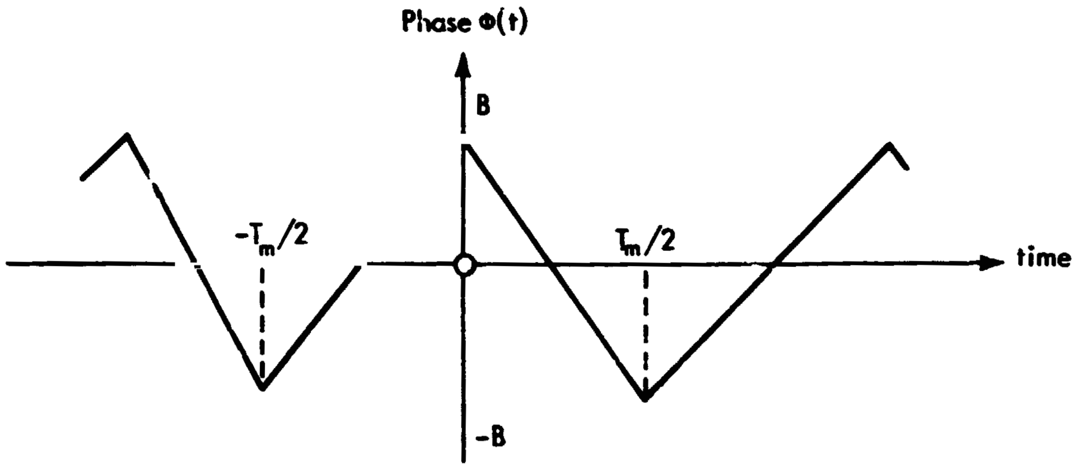


Figure B3. Triangular Wave Modulating Function

$$C_n = \frac{1}{jT_m\left(\frac{4B}{T_m} - \frac{2\pi n}{T_m}\right)}\left[e^{j(4B/T_m\, t + B - n\omega_m t)}\right]\Bigg|_{-T_m/2}^{0}$$

$$+ \frac{1}{-jT_m\left(\frac{4B}{T_m} + \frac{2\pi n}{T_m}\right)}\left[e^{j(-4B/T_m\, t + B - n\omega_m t)}\right]\Bigg|_{0}^{T_m/2} \qquad \text{(B16)}$$

$$C_n = \frac{-j}{4B - 2\pi n}\left[e^{jB} - e^{j(n\pi - B)}\right]$$

$$+ \frac{j}{4B + 2\pi n}\left[e^{-j(n\pi + B)} - e^{jB}\right] \qquad \text{(B17)}$$

using $e^{\pm jx} = \cos x \pm j \sin x$

$$C_n = \frac{-j}{4B - 2\pi n}\left[\cos B + j \sin B - \cos(n\pi - B) - j \sin(n\pi - B)\right]$$

$$+ \frac{j}{4B + 2\pi n}\left[\cos(n\pi + B) - j \sin(n\pi + B) - \cos B - j \sin B\right] \qquad \text{(B18)}$$

$$C_n = \frac{1}{4B - 2\pi n}\left[(\sin B - \sin(n\pi - B)) + j(\cos(n\pi - B) - \cos B)\right]$$

$$+ \frac{1}{4B + 2\pi n}\left[(\sin(n\pi + B) + \sin B) + j(\cos(n\pi + B) - \cos B)\right] \quad \text{(B19)}$$

using

$$\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$$

$$\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$$

$$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$$

$$C_n = \frac{1}{n\pi - 2B}\left[\sin \frac{1}{2}(n\pi - 2B) \cos \frac{1}{2}(n\pi) + j \sin \frac{1}{2}(n\pi) \sin \frac{1}{2}(n\pi - 2B)\right]$$

$$+ \frac{1}{n\pi + 2B}\left[\sin \frac{1}{2}(n\pi + 2B) \cos \frac{1}{2}(n\pi) - j \sin \frac{1}{2}(n\pi + 2B) \sin \frac{1}{2}(n\pi)\right] \quad \text{(B20)}$$

The final expression then becomes

$$|C_n| = \left| \left[\frac{\sin\left(\frac{n\pi}{2} - B\right)}{n\pi - 2B} + \frac{\sin\left(\frac{n\pi}{2} + B\right)}{n\pi + 2B}\right] \cos \frac{n\pi}{2} \right.$$

$$\left. + j \left[\frac{\sin\left(\frac{n\pi}{2} - B\right)}{n\pi - 2B} - \frac{\sin\left(\frac{n\pi}{2} + B\right)}{n\pi + 2B}\right] \sin \frac{n\pi}{2} \right| \quad \text{(B21)}$$

The results of this equation are tabulated in Table E2 and the corresponding spectrum pictures are shown in Figure B4.

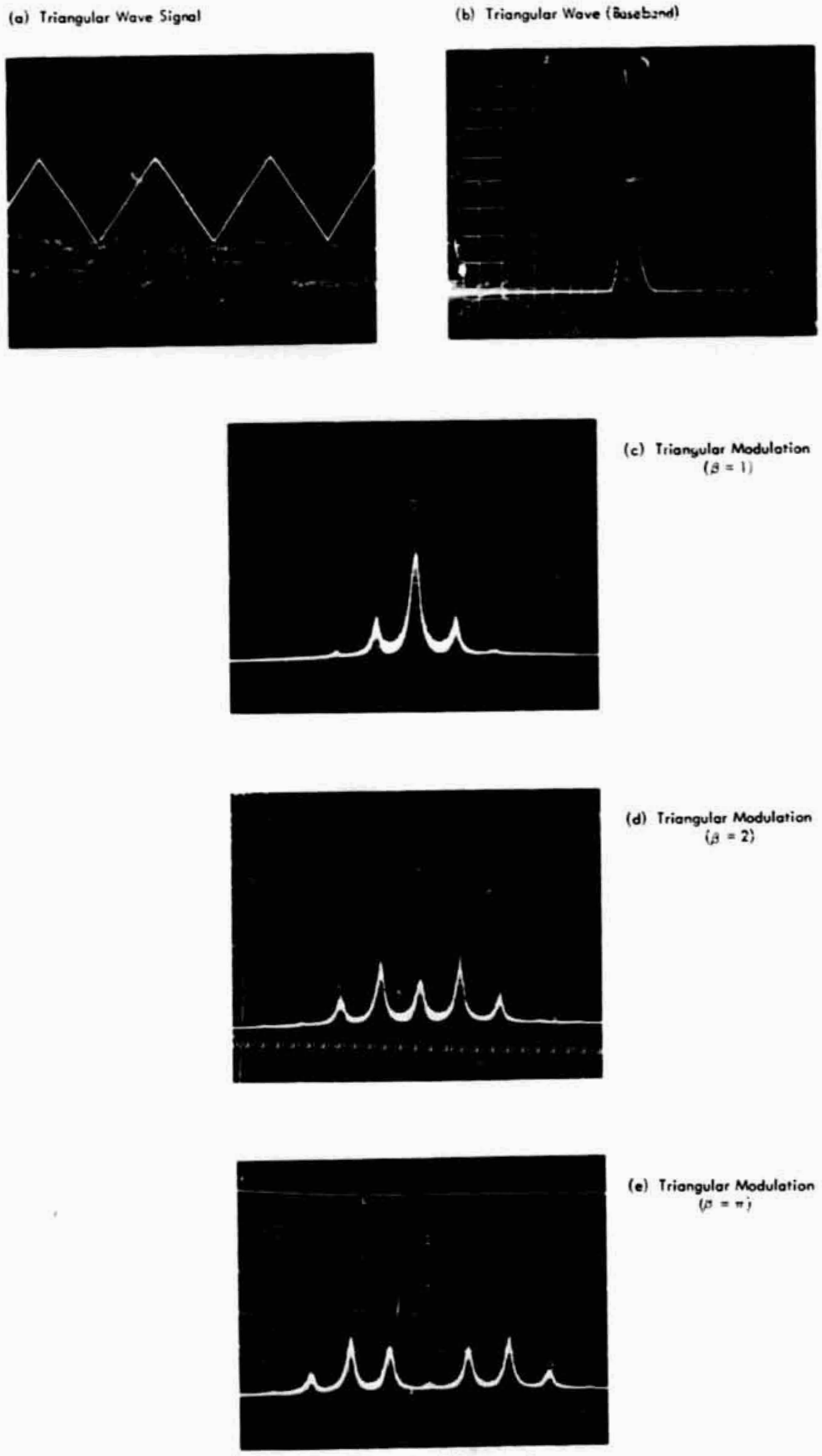


Figure B4. Triangular Modulation Spectrums

TABLE B2
Triangular Wave – Phase Modulation
Modulation Index vs. Sideband Amplitude

BETA (RAD)	C0 (DB)	C1 (DB)	C2 (DB)	C3 (DB)	C4 (DB)	C5 (DB)	C0/C1 (DB)
0.10	-0.01	-27.85	-59.87	-46.97	-71.86	-55.85	27.84
0.20	-0.06	-21.86	-47.86	-41.07	-59.91	-49.95	21.80
0.30	-0.13	-18.38	-40.85	-37.75	-52.94	-46.65	18.25
0.40	-0.23	-15.94	-35.89	-35.54	-48.04	-44.46	15.70
0.50	-0.36	-14.07	-32.07	-33.99	-44.28	-42.92	13.71
0.60	-0.53	-12.58	-28.96	-32.89	-41.25	-41.86	12.05
0.70	-0.72	-11.35	-26.36	-32.16	-38.73	-41.16	10.63
0.80	-0.95	-10.31	-24.13	-31.75	-36.61	-40.79	9.37
0.90	-1.21	-9.43	-22.18	-31.65	-34.78	-40.74	8.23
1.00	-1.50	-8.68	-20.46	-31.88	-33.20	-41.01	7.18
1.10	-1.83	-8.03	-18.92	-32.48	-31.83	-41.67	6.20
1.20	-2.19	-7.47	-17.54	-33.58	-30.63	-42.83	5.27
1.30	-2.60	-6.99	-16.30	-35.12	-29.59	-44.74	4.39
1.40	-3.05	-6.58	-15.17	-38.60	-28.69	-48.00	3.53
1.50	-3.54	-6.23	-14.14	-45.49	-27.92	-54.98	2.69
1.60	-4.09	-5.94	-13.20	-52.47	-27.27	-62.02	1.86
1.70	-4.68	-5.71	-12.34	-38.91	-26.73	-48.57	1.03
1.80	-5.34	-5.53	-11.55	-33.32	-26.31	-43.10	0.19
1.90	-6.05	-5.39	-10.84	-29.62	-26.00	-39.51	-0.66
2.00	-6.85	-5.30	-10.18	-26.80	-25.80	-36.81	-1.54
2.10	-7.72	-5.26	-9.58	-24.50	-25.73	-34.65	-2.46
2.20	-8.69	-5.26	-9.03	-22.55	-25.79	-32.85	-3.43
2.30	-9.78	-5.31	-8.53	-20.86	-25.99	-31.32	-4.48
2.40	-11.01	-5.39	-8.08	-19.36	-26.36	-29.99	-5.62
2.50	-12.42	-5.52	-7.67	-18.05	[illegible]	-28.84	-6.90
2.60	-14.05	-5.70	-7.31	-16.82	-27.75	-27.84	-8.36
2.70	-16.01	-5.91	-6.99	-15.72	-28.91	-26.96	-10.10
2.80	-18.44	-6.18	-6.70	-14.72	-30.56	-26.20	-12.27
2.90	-21.67	-6.49	-6.46	-13.80	-33.02	-25.54	-15.18
3.00	-26.55	-6.85	-6.25	-12.96	-37.15	-24.98	-19.70
3.10	-37.45	-7.26	-6.08	-12.19	-47.31	-24.51	-30.19
3.20	-34.78	-7.72	-5.95	-11.47	-43.89	-24.14	-27.06
3.30	-26.41	-8.25	-5.85	-10.81	-34.79	-23.86	-18.16
3.40	-22.48	-8.84	-5.78	-10.21	-30.14	-23.66	-13.64
3.50	-19.98	-9.50	-5.75	-9.65	-26.92	-23.57	-10.48